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ol. XXVIII January, 1941 No. 1

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e advancement of Letters, Science, and Art

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THREE PUBLIC LECTURES ON SCIENTIFIC SUBJECTS[1]

I

A MATHEMATICAL APPROACH TO ETHICS[2]

IN January, 1932, when I had the privilege of lecturing at the Rice Institute, my topic was that of quantitative aesthetics. In the first of the lectures which I am giving now, I would like to call your attention to similar possibilities in the field of ethics. It is an especial pleasure for me to be with you again; and I can only hope that you will find your confidence justified, at least to the extent that the three very diverse subjects considered will be novel and interesting. Only in the last of these will any mathematical technique be involved.

Since the time of the German philosopher, Immanuel Kant, it has been clear that, for certain purposes, philosophic thought may be treated separately in its logical, aesthetic, and ethical aspects, concerned respectively with the true, the beautiful, and the good.

In the last century logic has developed into an independent discipline—the edifice of syllogistic thought—of which all of mathematics appears as the grandiose superstructure.

The concept of "aesthetic measure" which I laid before you in 1932[3] made possible a more or less mathematical treatment of aesthetics giving promise of taking the subject of analytical aesthetics out of the domain of philosophic speculation into the region of common sense thought. The question thus presents itself almost irresistibly to the mind: Is not a similar treatment of analytic ethics possible? My aim here is to show that such a program seems to be feasible.

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[1] Delivered at the Rice Institute, March 6, 7, and 8, 1940, by George David Birkhoff, Ph.D., Sc.D., LL.D., Perkins Professor of Mathematics at Harvard University.

[2] Many of the ideas presented in this lecture were first given by the author in a lecture entitled "A Program for Quantitative Ethics," delivered at the University of Washington under the auspices of the Walker-Ames Foundation, July 19, 1939.

To most mathematicians the tendency towards increasing mathematization in these three fundamental aspects of philosophic thought—logic, aesthetics, and ethics—is only what was to be expected; for they are likely to agree with the dictum of the great French philosopher and mathematician, René Descartes, *omnia apud me mathematica fiunt*—with me everything turns into mathematics!

Even in early Greek times the philosopher Pythagoras tried to bring mathematical order into the ethical field by asserting that justice is represented by a square number. This must be looked upon as a mystical conjecture of real importance for ethics. Similarly Plato and Aristotle were always desirous of showing the close relationship of the good and the beautiful, if not their essential identity; and they regarded the beautiful as characterized by unity in variety. Thus, there has always been observable in ethics, as well as in aesthetics, a tendency towards quantitative formulation. The supreme goal of the *summum bonum* or highest good, adopted by the Greeks, is suggestive of this; and the modern utilitarian principle of "the greatest good of the greatest number" reveals still more clearly the same tendency.

A very interesting analogy between aesthetics and ethics is the following. Individuals of so-called artistic temperament often look upon their personal experiences as a succession of aesthetic adventures from which they try to extract the greatest possible enjoyment. Similarly, persons of predominantly moralistic type strive for a maximum of moral satisfaction by making in their daily lives such ethical decisions as will best promote the material and spiritual well-being of their fellows.

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[3] See my lectures, "A Mathematical Theory of Aesthetics and its Application to Poetry and Music," delivered at the Rice Institute in January, 1932, published in the *Rice Institute Pamphlet*, Vol. XIX, No. 3, July, 1932; and also my book, *Aesthetic Measure* (Cambridge: Harvard University Press, 1933).

Just as the analysis of experience from the aesthetic point of view yields the concept of "aesthetic measure"—the ratio of aesthetic reward to effort of attention—as basic in the evaluation of aesthetic pleasure, so the consideration of experience in its ethical aspects leads to an analogous concept of "ethical measure"—the amount of moral satisfaction based on good accomplished.

The simple ethical formula evidently suggested is:

$$M\ (\textit{ethical measure}) = G\ (\textit{total good achieved}).$$

From this point of view the ethically-minded person[4] endeavors always to select that one of the possible courses of action which *maximizes* the ethical measure G, just as the aesthetically-minded person continually compares aesthetic objects and prefers those which maximize the aesthetic measure O/C[5]. The utilitarian calculus of Jeremy Bentham represents a suggestive semi-philosophical attempt in the same direction.[6]

Let us consider a little more in detail this general parallelism between the aesthetic and ethical domains. In order to do this the use of parallel columns is convenient.

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[4] Or corporate body or state.

[5] O = order, C = complexity.

[6] In this connection, Mr. P. A. Samuelson of the Society of Fellows of Harvard University calls my attention to F. Y. Southworth's very interesting volume on *Mathematical Psychics* (1881).

Aesthetics

Some of the principal aesthetic 'factors' are: (+, of positive type) repetition, similarity, contrast, balance, sequence, centers of interest or repose; (−, of negative type) complexity, ambiguity, undue repetition, unnecessary imperfection. These factors enter into the terms O and C of the aesthetic formula,

$$M = O/C$$

The factors involved in the order O may be divided into formal and connotative elements of order, while the complexity C is formal. Only the formal type of elements in O admits of quantitative treatment.

In aesthetics, objects of a definite class are to be compared in regard to their relative aesthetic measures M. Such classes are of extraordinary variety. The theory of aesthetic measure is best exemplified by certain simple formal visual

Ethics

Some of the principal ethical 'factors' are: (+, of positive type) material good, sensuous enjoyment, happiness, intellectual and spiritual achievement; (−, of negative type) material waste and destruction, pain, sorrow, intellectual and spiritual deterioration. These enter into the term G of the ethical formula,

$$M = G$$

The factors involved in the good, G, may be divided into the material and the immaterial elements of the good. Only the material type of elements admits of quantitative treatment by the formula.

In ethics, each single definite problem is to be considered by itself, and the possible solutions are compared as to their ethical measures, M. These problems are also of extraordinary variety. The main interest in ethics is provided and auditory fields, provided by art rather than by nature.

Artists, connoisseurs, and critics of all kinds are considered to be especially competent judges in their special aesthetic fields. But the aggregate opinion of ordinary lay observers plays a vital rôle.

Aesthetic tastes vary from one individual to another, and are relative to the period and culture concerned. Nevertheless there is a certain grand parallelism to be discerned, due to the presence of certain absolute elements of order, as, for instance, rhythm in music. Cultivated human beings are generally able to understand and appreciate aesthetic objects of all kinds and periods.

Finally, the main phases in the history of aesthetic ideas and literary criticism of special artistic forms can be concisely interpreted by use of the concept of aesthetic measure.

by problems arising in practice rather than by artificial problems.

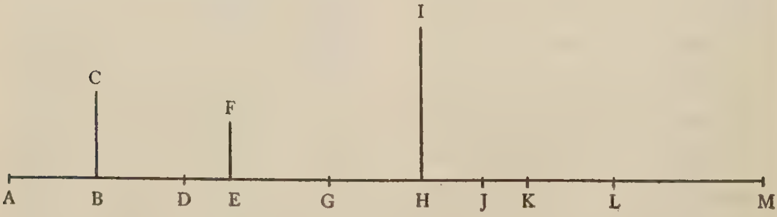
Religious leaders, statesmen, judges, and the socially elect are regarded as the best judges in their several ethical fields. But the general intuitive opinion of mankind often has decisive weight.

Ethical values and ideals vary in a similar manner. Nevertheless, there are always to be found certain absolute elements of the good as, for instance, bravery and loyalty in their socially validated forms. Careful study of the development of such specific forms serves to explain them acceptably to men everywhere as varied manifestations of these absolute elements of the good.

Similarly, the main phases in the history of ethical ideas and of their many special social manifestations admit of concise interpretation through the concept of ethical measure.

Having called attention to this significant general parallelism between aesthetics and ethics, I propose to consider some specific problems which illustrate how the concept of ethical measure can be used. I make no apologies for the simple character of the ideas involved, since it is inevitable that the initial results obtained be rudimentary. As possibly suggestive in this connection, it may be recalled that the first classification of matter as solid, liquid, or gaseous provided a crude trifurcation of nature, which ultimately led to the mathematical theories of elasticity and hydrodynamics.

Problem I. A bus driver regularly takes passengers from the starting point *A* to their destinations along the main road from *A* to *M* and along certain side roads on one side of the main road. The majority of the passengers live along the main road, and the side roads are short. The driver wishes to be as accommodating as possible and to give all the passengers equal consideration. In what order should he take the passengers to their destinations?



His decision is always to deliver the passengers in the natural order of going from *A* to *M* along the main road. To justify this decision he might argue as follows:

Suppose first that all the passengers on some trip wished to alight at points on the main route, as not infrequently was the case. If he took them to their destinations in other than the natural order, the series of passengers (as a series) would be less quickly delivered one by one than otherwise, i.e., the first passenger would alight later, the second passenger also, etc. Since all the passengers are to be treated as well as possible, this would be regarded as extremely objectionable by them. But in the event that some of the passengers wish to alight along the short side roads, the additional times required are very small and in the driver's judgment do not need to be considered. Hence he finds that the delivery of passengers should always be as stated.

Let us attempt to formalize this simple reasoning. The underlying good here, G, may be regarded here as negative $(-)$, if we reckon upon the unrealizable good of immediate delivery of the passengers as the neutral point (0) from which the reckoning starts. Thus we write

$$G = -(\text{sum of all the trip-durations for the passengers}).$$

The possible solutions to be considered are the various ways of taking the passengers to their destinations.

The two basic assumptions of the driver are almost but not entirely in agreement with this definition of G; they are: (1) the individual trip-durations along the main road are to be diminished as far as possible; (2) the trip-durations along the side roads need not be considered. On this basis his decision is obviously as stated and in general will maximize the good, G, as just defined.

However, there might occasionally arise situations in which this solution was not actually the best one by the formula written above. Suppose, for example, there were six passengers, one to be delivered at C, and five at D, with equal distances AB, BC, and BD (see the figure above). Clearly the 'best' solution in this exceptional case would be to deliver the five passengers at D along the main road, and then to return along the main road and deliver the remaining passenger at C. In fact, if the driver follows his general rule, we have

$$G = -22a,$$

where a stands for the time required for the bus to go over any one of the equal distances, while if the driver were to deliver his passengers in the reverse order, we would have

$$G = -14a,$$

so that $8a$ units of time would be thereby saved to the passengers.

Nevertheless, the driver decides to deliver the passengers in the usual way. In doing so he goes directly against a perfectly natural postulate referred to above, namely, that if he can shorten (or in no case lengthen) the trip-durations of the successive passengers, he certainly should do so.

Obviously the naturalness and uniformity of the solution adopted by him operates as an important factor in its favor. For the rule of procedure chosen by the driver is readily understood by the passengers and any modification of it in the direction of increased complication might lead to dissatisfaction, especially because the time-schedules of the trip would become even more unpredictable.

Thus we are led to realize that there are instances in which the simplicity and elegance of the solution of an ethical problem must itself be regarded as one of the imponderable elements of the good which enter into G.

There is a kind of counterpart to this phenomenon in the aesthetic field: Apparently the intuitive aesthetic judgment tends through an inner necessity to prefer formally simple elements of order in the aesthetic object.

It would be easy to propose other allied problems in this field of the ethics of procedure. In fact, the last two problems here considered (*V* and *VI*), dealing briefly with the proportionate representation of the states in the House of Representatives and with the question of preferential balloting, are both clearly of the procedural type. All such problems are characterized by the fact that they involve the determination of the most equitable procedure by an agent to whom the choice of procedure is personally indifferent, while the good, *G*, in question is a material good.

Our second problem is intended to present a very different type of ethical situation which is of high significance, and which involves both material and immaterial elements of the good. Though presented in a specialized form, I believe that the problem selected embodies a situation characteristic of critical moments in the lives of many human beings—moments when the choice must be made between material good with attendant failure in loyalty, on the one hand, or the sacrifice of this material gain with preservation of loyalty, on the other. As has been indicated previously, a complete quantitative treatment cannot be hoped for in such a problem.

Problem II. One or the other of two friends of long standing, *A* and *B*, is to be advanced to an opening in the organization in which they hold positions of the same rank. *A* happens to learn that the actual selection will hinge upon the judgment of a certain person *L* belonging to the same organization. Ought *A* to pass this information on to *B*?

The answer of course is that in the circumstances stated *A* ought to inform his friend *B*.

A's reasons for this decision might be formalized as follows: The material goods g_A and g_B which will accrue to him or to his friend through such an advancement are the same: $g_A = g_B = g$. If *A* informs *B*, the immaterial good of his friendship, f, with A is retained. Therefore, we have simply

$$M = g.$$

On the other hand, if A does not tell B what he has learned, the friendship between them is destroyed, even if B never learns of A's unfriendly act; and so we have

$$M = g - f.$$

Since g exceeds $g - f$, A ought to tell B, although he realizes that by doing so he gives up a definite personal advantage. In the above reckoning the unfavorable effect upon A's character of not informing B is intentionally disregarded although it might really be the most important consideration of all.

A's decision to pass on the information to B is here assumed to be made on the utilitarian basis. On a hedonistic basis, A might conclude that if he fails to inform B, then

$$M = g - f,$$

since he will be certain to win L's special favor, whereas, in the contrary case,

$$M = \frac{g}{2},$$

inasmuch as he would then only have an equal chance with B. In this event, he would have to balance the prospect of material advancement against his friendship with B.

Again, according to the extent that A believes himself inferior to B, he will feel that his chances are lessened by telling B. If A is a loyal friend, however, he will not be moved from his decision by such thoughts.

The basic hypothesis has been made here that the information about L is of legitimate practical advantage to A

and *B*. It is also assumed that the friendship between *A* and *B* is a sincere one, founded upon mutual esteem. For clearly if there were no real friendship, *A* would not be under any obligation to inform *B*, any more than he would consider it an obligation on *B*'s part to tell him. Of course if *A* believes that *B* would not tell him if circumstances were reversed, or that *B* would employ unfair or unscrupulous tactics to gain *L*'s favor, the bond of friendship between them is already weak; and so the situation would not be the one envisaged in the problem under consideration.

A somewhat similar type of problem, also not infrequently exemplified in human experience, is the following:

Problem III. Two men, *A* and *B*, among six, *A*, *B*, *C*, *D*, *E*, *F* in control of a certain business, have orally agreed to exchange all relevant information before entering into any arrangement with the others. *A* and *B* do this in order to protect their interests in the business. *A* is approached confidentially by *C*, *D*, *E*, and *F*, and asked if he will concur in a vote giving him important special privileges which are to be withheld from *B*. Actually *A* does not feel he is entitled to these special privileges any more than *B* is. How ought *A* to act?

The ethical course for *A* to follow is clearly to refuse to connive with *C*, *D*, *E*, and *F*. He should further inform *C*, *D*, *E*, and *F* that in his opinion to do otherwise would not be fair to *B*.

If *A* acts in this manner we may write

$$M=0,$$

meaning thereby that the *status quo ante* is not altered. If *A* consents to their proposal, we may write for *A*

$$M=g_A-f-e_B,$$

meaning that A gains the privileges mentioned (g_A), loses B's friendship (f), and possibly incurs B's positive enmity (e_B), fraught with danger to him—for instance, the enmity of B might lead to his loss of a valuable reputation for business integrity. Here we find in G two elements of mainly material nature (g_A, e_B) and one of immaterial nature (f).

The two preceding problems have been taken from the field of social ethics. It is of interest that similar problems can be drawn from the field of international ethics. In the problem about to be stated there is no intention to parallel closely any actual problem. The intention is rather to suggest that there may exist somewhat analogous problems which admit of clarification when approached from the point of view of ethical measure.

Problem IV. As the result of a war, B has lost a colony C to the nation A. This colony C has subsequently been given nearly complete independence by A. This action leaves C well satisfied with her status and favorably disposed towards A. However, B has an economic need for her former colony C, by reason of lack of raw materials which C had formerly supplied; and for this and other more political reasons, B demands the cession of C back to her by A. How is A to reply to the demand?

A reasonable analysis on A's part might be the following: A concludes that to return the colony C would not only be objectionable to C but extremely detrimental to A's international standing and prestige as a concession under duress. Furthermore, A feels that if she did agree to B's demand, other similar demands reinforced by further military threats would soon follow. Thus A (and C) might write in the event of return of C to B

$$M = g_B - h_{A,C}$$

(where $g_B =$ material good to B; $h_{A,C} =$ material and immaterial harm to A and C) and, in the contrary case,

$$M = 0,$$

since there is no reason to believe that the prospect of ultimate war is effectively lessened. Hence A and C would have to balance B's good against their own harm; and so A would almost certainly refuse to cede C back to B.

From B's standpoint, however, the analysis in the case of cession would more correctly be

$$M = g_B + p,$$

(where $p =$ good of peace), since B would not admit that A or C would suffer much economic loss thereby, nor that there could be enduring peace without cession; and B's analysis in the contrary case would be

$$M = 0.$$

Thus the balance in favor of cession is $g_B + p$ in B's estimate, and at least $h_{A,C} - g_B$ against cession in A and C's estimate. Thus there is a very serious conflict of ethical judgment. Such a situation naturally raises the question of possible compromise.

In this and similar cases of apparent conflict in ethical judgments the thorough exploration of all possibilities of compromise is absolutely essential.

The following is a suggestion of a possibility of such a compromise in this particular case: A notifies B and C that in recognition of B's economic needs and of her claims, she will henceforth not accept from her colony C any more favorable trade status than C accords to B.

There is then the possibility that despite A's refusal of B's demand for the return of C to her, B can recover a substantial portion of her former trade with C. Thus one might write, in behalf of all the three parties A, B, and C, a formula such as

$$M = \tfrac{1}{2}g_B - \tfrac{1}{4}g_A + \tfrac{1}{2}p,$$

in the event of such a compromise, as against $M=0$ if the *status quo ante* is preserved. The sole loss for A would be loss of an estimated quarter of B's trade with her ($\frac{1}{4}g_A$); B would recover an estimated half of her former trade with C ($\frac{1}{2}g_B$) without loss to C; and it might be that the resultant improvement in the friendliness of relations between A and B would increase the likelihood of a permanent peace and so slow down the expensive armament race between A and B ($\frac{1}{2}p$).

The question of compromise is extremely important in many ethical problems. Is it reasonable to suppose that such compromises can generally be reached? In this connection I recall a conversation with Dean Roscoe Pound and Count Korzybski some years ago. Count Korzybski had expressed the opinion that many conflicts of points of view had their origin mainly in misunderstandings as to the meaning of terms, so that the conflict would disappear as soon as these meanings were agreed upon. I replied that in many disputes the situation resembled rather that arising between two boys contending for a single piece of pie; and Dean Pound was inclined to agree with me. In the tragic condition of the world today, the suggestion might be made that if the division of the single piece of pie into two equal pieces were made (a reasonable compromise), both boys could be induced to accept their portion!

Our last two problems, like the first, fall in the field of ethical procedure, and are interesting in showing that a certain amount of technical mathematical consideration may be necessary. The first of these is that of the equitable apportionment of Representatives in our Congress to the several states in accordance with the constitutional provision to that effect:

Problem V.[7] The set of states is to elect biennially an assigned total number of Representatives to Congress, at least one from each state; and, in accordance with the Constitution of the United States, the numbers of Representatives allotted to the several states are to be as nearly proportional as possible to the populations of the respective states. What is the best method of the apportionment of Representatives to the various states?

It is first of all to be observed that various intuitively reasonable postulates may be formulated, such as, for instance, the two following: Of two unequal states *A* and *B*, the one with greater population should have at least as many Representatives as the other; every state should receive at least the integral part of the exact (fractional) number which it is ideally entitled to.

These two postulates were satisfied by the former Vinton method of apportionment which may be described as follows. The theoretical size of a congressional district (i.e., its population) is first calculated, and on this basis the exact number of Representatives (not in general an integer) for each state is determined. Each state then receives at least as many Representatives as the integral parts of these numbers; and one additional Representative is assigned to the states in the order of decreasing fractional parts of these numbers until the required number of congressional Representatives is reached. Thus, if there were three states with the calculated numbers 3.72, 2.41, 1.87, with a corresponding total of eight Representatives, there would first be assigned respectively, 3, 2, and 1 Representatives, leaving two still to be assigned. These would then go to the third and first states with the larger fractional parts, (.87 and .72 respectively) so that the final assignments by the Vinton method are 4, 2, and 2 respectively.

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[7] For the mathematical conclusions accepted below without analysis see E. V. Huntington's article "On the Method of Equal Proportions," *Trans. Amer. Math. Soc.*, pp., 85-110, vol. 30, (1930).

It is significant that this very simple method was accepted until it led to the "Alabama paradox," exemplified when an increase in the total number of Representatives from 299 to 300 led to an actual decrease in Alabama's quota of Representatives! Here there was violated another very simple and natural postulate: If the assigned total of Representatives for all the states is increased, the assignment to each of them should certainly not be diminished. This outcome was manifestly unreasonable from the political point of view, and led to the use of Willcox's so-called "method of major fractions" instead of the Vinton method.

Willcox's method is very simple in statement and is at once seen to avoid the "Alabama paradox." It may be formulated as follows: Determine as before the number of Representatives for each state, and assign to each state the next lesser integral number if the fractional part is less than one-half, and the next greater if it exceeds one-half (i.e., is a *major fraction*). Then, at least if this rule yields the desired total number of Representatives, the assignment will be that designated by the Willcox method. If it does not do so, increase or decrease the (theoretical) size of a "congressional district" until the correct total number is secured. This allocation of Representatives will yield the proper result, according to the Willcox method.

More recently, Huntington has proposed his method of equal apportionment which not only avoids the Alabama paradox but has two additional formal advantages over the Willcox method: (1) it *automatically* assigns at least one Representative to every state, in accordance with the constitutional requirement, as the Willcox method may fail to do; (2) the Huntington method makes no distinction between the mathematically dual notions of "congressional district" and "individual share" while the Willcox method arbitrarily prefers the latter.*

Thus here again considerations of simplicity and elegance enter in determining the relative merits of the two methods of apportionment.

In Willcox's method, from our standpoint of "ethical" measure, it is not hard to show that there is an underlying ethical measure for any two states

$$M = -\left|\frac{C_1}{P_1} - \frac{C_2}{P_2}\right|$$

where P_1 and P_2 are the populations of the two states, and C_1 and C_2 are the proposed numbers of congressional seats for these states. In the unnamed dual method we have similarly

$$M = -\left|\frac{P_1}{C_1} - \frac{P_2}{C_2}\right|.$$

The corresponding choice of M for Huntington's method of equal apportionment is

$$M = -\left| \log\frac{P_1}{C_1} - \log\frac{P_2}{C_2}\right|.$$

*According to Huntington (Congressional Record, April 28, 1941) the test of equal proportions is to be stated as follows: "A proposed transfer of a seat from one state to another state should be made when and only when the percentage inequality between the congressional districts in the two states is reduced by the transfer." The words "congressional districts" may be replaced by "individual shares" if one so desires, since Huntington's method of apportionment is self-dual.

All three methods fulfill the following further important and natural requirement: The assignment should be made in such wise that it cannot be improved between any pair of states by transferring a Representative from one state to the other.

The second postulate mentioned above as satisfied by the Vinton method is not satisfied by any of these three methods.

It seems to be abundantly borne out in this and other examples that apparently justifiable postulates are often mutually contradictory, so that a choice has to be made between them.

Two further remarks of general import may well be made. If the populations of the states were to remain nearly fixed for long periods of time, injustices might be regularly inflicted upon certain of them, so that some further modification of the method of apportionment mentioned might become desirable. In the second place, from the standpoint of ethical measure, it would be of interest to investigate other possibilities. For example it might be our goal to minimize the largest injustice of underrepresentation for any state, then the next largest injustice for some other underrepresented state, etc., and we might ask what would then be the corresponding method of apportionment. Here we might adopt the specific measure of injustice suggested by the value of M given above for the method of equal apportionment of Huntington.

We turn now to a very brief consideration of our final problem of preferential balloting:

Problem VI. A group is required to elect one of a number of candidates to an office on the basis of the relative preferences of the group. How should the successful candidate be determined from the ordering of names on the various ballots?

Here, too, various natural postulates obviously apply, such, for example, as the following: If one of the candidates, A, is named in higher position than another, B, throughout (i.e., A has more first positions, more first and second positions combined, etc.,) then A is to be chosen in preference to B. By the aid of this principle alone the choice is generally narrowed down to a few individuals.

The *simplest* rule of procedure perhaps is to rate the first position as 1, the second as 2, etc., and to add the ratings for each candidate. The candidate with the least total is then taken to be elected[8]. In this case we may write for each candidate

$$M = k - (\text{sum of positions obtained}),$$

where k is the number of members in the voting group. Thus, if one of the k candidates receives all the first choices, we have $M = 0$, the ideal case; otherwise M is negative and is smallest for the least sum of indices of position.

Another possible definition of M would be

$$M = 1 - \sqrt[k]{\text{product of positions obtained.}}$$

These two methods would be related much as the arithmetic and geometric means are. However the first (usual) method has the advantage of being simpler to apply in practice.

It is clear that a politically-minded person could manipulate his ballot in favor of special candidates by not indicating his real preferences. However, such action would constitute a serious misuse of the preferential form of balloting.

This problem of preferential balloting has not been treated as thoroughly as it should be.

In my judgment it would be a very constructive program for analytic ethics, to catalogue systematically various significant

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[8] The case of a tie must be otherwise decided of course.

problems in the three fields of procedural ethics, social ethics, and international ethics, and to classify the main types of solutions on the basis of the formula for ethical measure.

This has been attempted to a very rudimentary extent in the six special problems above.

More specifically, social customs and systems of law and of religion contain a vast mass of ethical data, embodying the accepted ethical solutions of innumerable practical problems of analytic ethics; and the inductive method can generally be applied to treat new problems when they arise. In so far as these solutions are not purely empirical, they could be codified by means of the ethical formula. Such a codification would list and classify the very extensive variety of ethical intuitions (postulates), in part the cause of, and in part the result of, specific social interactions. There is little doubt of the basic rôle which the sentiments of love, goodwill, loyalty, and other feelings of sensuous, aesthetic, or intellectual type play in such intuitions. These provide a substratum of absolute elements, of which the specific manifestation depends on the particular culture and period concerned.

Another useful service of such a program might be to treat the extremely interesting history of ethical ideas by use of the same ethical formula. Thus the early Greek conception of ethical behavior as directed towards the attainment of the *summum bonum* is evidently in consonance with the ethical formula. The customary threefold division of ethical theories into those of hedonistic (or egoistic) type, of utilitarian (or universalistic) type, and of altruistic type is immediately explained in the same way; for if G_I denotes the good of an individual and G_F the good of his fellows, then the three types of ethical theory correspond to the respective formulas: $M=G_I$, $M=G_I+G_F$, $M=G_F$ respectively. It is hardly necessary to say that in promulgating a theory which is (supposedly) of the first or last non-utilitarian type, it is frequently necessary to rob Peter to pay Paul!

Many ethical theorists have tended to take the good and the pleasurable as synonymous; thus, according to Bentham, pleasures differ quantitatively but not qualitatively. From our point of view, this is a necessary assumption if all the constituents of the good in G are looked upon as comparable quantities, as required by the ethical formula.

Some have regarded the striving for perfection as supremely important, thereby emphasizing the achievement of potential good as the final goal; this reaches far into the domain of the *qualitative* application of the ethical formula.

Still others, like Kant, insist upon the dominating rôle of the sense of duty as the "categorical imperative." This validates the innumerable ethical intuitions on which concrete decisions concerning the immaterial good must always depend. Through the sense of duty we feel that it is possible to distinguish clearly between right and wrong, independently of our particular backgrounds, although careful analysis reveals that this independence is by no means complete. In fact the formalization of such intuitions, combined with the use of the general ethical formula, leads to the analytic solution of ethical problems by means of reasoning—a point of view going back to Socrates, Plato, and Aristotle.

There is a further reason why the systematic codification of ethical notions might be of genuine service. Ordinary language provides a vast storehouse of convenient symbols, which (as has been recently emphasized) often bring together under one name a number of quite different entities. For example, we speak of "fatigue" with a good deal of certainty. But what is fatigue? There are specific conditions of fatigue of the muscles, of special nerves, etc; but what have they in common? Careful experimentation, systematic analysis, and detailed classification are necessary for the proper elucidation of this question. This has indeed been accomplished recently by Professor L. J. Henderson and his colleagues in the Harvard Fatigue Laboratory; and the final upshot is that "fatigue" designates not one thing but many, grouped together largely because of intellectual convenience. Of course the notion of fatigue has no immediate ethical import.

Now many of the terms used constantly in ethical discussions have an even less definite meaning, and frequently provide a convenient emotional support for ethical or unethical action. Certain of these general terms, such as "wisdom" and "justice" seem to be mainly constructive in their effect, but others, like "racial superiority," for example, are positively destructive and dangerous, unless their various meanings have been made very specific. For instance, in speaking of racial superiority, which of the qualities listed below do we regard as really characteristic? Physical prowess and beauty; racial purity; descent from divine ancestors; intellectual capacity and achievement; aesthetic sensibility and artistic creativeness; unselfish idealism; unlimited devotion to the state; economic efficiency; military might; high potentiality of further development? Evidently there are many consistent points of view as to what constitutes "racial superiority"; and so wherever the idea is used it needs to be properly defined and accurately applied in the selected sense.

Our thought here is akin to that of Korzybski, that when human beings realize that certain important general terms have a variety of distinct meanings, the effect of this realization is definitely prophylactic against misunderstanding, prejudice, and intolerance. If the clarification of such important but multiple-valued ideas is not effected soon, the present tragic confusion among men may end in the destruction of civilization.

Such is the general program for ethics to which I desire to direct attention. It is analogous in many respects to that which I have attempted to carry out provisionally in aesthetics. The program involves the introduction of elementary quantitative ideas based on a simple formula for "ethical measure" in order to clarify and codify the vast ethical domain. Conceivably such a program might perform the same kind of useful service for ethics as elementary logic performs for mathematics, and grammar for language.

II

THE PRINCIPLE OF SUFFICIENT REASON

AN outstanding characteristic of present-day civilization is the extraordinary rapidity of scientific advance, accompanied by a veritable Babel of changing scientific theories. The ceaseless flux has produced confusion in the minds of men. Even philosophy is affected by the prevailing uncertainties, and many assert that its speculations are meaningless unless narrowly restrained to the mathematical and logical fields.

But philosophy has an answer ready for these detractors. For it has been Plato, Aristotle, Leibniz, and other philosophers who first emphasized the basic significance of mathematics and logic; and it may be reasonably conjectured that in the future, seers endowed with the requisite philosophic insight will again open up large vistas which the specialized scientist, mathematician, and logician are likely to miss. Only those are likely to contribute in this way who, like Plato, Aristotle, and Leibniz, have mastered the essence of contemporary scientific knowledge. Unfortunately the task of achieving this necessary synthesis is becoming a more and more difficult one.

It will be recalled that Plato had inscribed at the portals of his famous Academy, "Let no one ignorant of Geometry enter here." His whole philosophic system affirmed the supreme importance of mathematical thought and of abstract Ideas generally, of which the concrete instances to be found in the actual world were held to be merely inferior copies.

His contemporary, Aristotle, was the creator of formal logic.

The German mathematician and philosopher Leibniz has been regarded by some as Plato's true successor in modern times. He too, emphasized the fundamental rôle of abstract thought for philosophy, and foresaw the possibility of a logical calculus such as is realized in modern symbolic logic. He said, "My metaphysics is all mathematics" and even went so far as to declare that "The mathematicians have as much need of being philosophers as the philosophers of being mathematicians."

At the foundation of Leibniz's philosophy were two logical principles to which he attached the greatest importance: the Principle of Contradiction and the Principle of Sufficient Reason. Both seem to have been explicitly adopted by him as early as the year 1686. The first principle merely asserts that every proposition is either true or false—*tertium non datur*. Nearly everyone would admit without hesitation that this is a valid principle if not indeed a truism; and yet, strangely enough, its validity has been denied in the intuitionist logic of the Dutch mathematician Brouwer, who allows, as a kind of intermediate category, the proposition which admits neither of proof nor disproof. It is the second principle, in modified form, to which I wish to direct attention.

It is desirable to recall in the first place Leibniz's own formulation of the Principle of Sufficient Reason,[1] "that every true proposition, which is not known *per se*, has an *à priori* proof, or, that a reason can be given for every truth, or, as is commonly said, that nothing happens without a reason." Leibniz adds that "Arithmetic and Geometry do not need this Principle, but Physics and Mechanics do, and

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[1] The translations here used are those of Bertrand Russell, "A Critical Exposition of the Philosophy of Leibniz," (Cambridge, England, 1900).

Archimedes employed it." Furthermore he used the same principle for metaphysical purposes as the following remarkable conclusion, obviously based on this Principle, shows: "When two things which cannot both be together, are equally good; and neither in themselves nor by their combination with other things, has the one any advantage over the other: God will produce neither of them."

My primary purpose will be to show how a properly formulated Principle of Sufficient Reason plays a fundamental rôle in scientific thought and, furthermore, is to be regarded as of the greatest suggestiveness from the philosophic point of view.[2]

In the preceding lecture I pointed out that three branches of philosophy, namely Logic, Aesthetics, and Ethics, fall more and more under the sway of mathematical methods. Today I would make a similar claim that the other great branch of philosophy, Metaphysics, in so far as it possesses a substantial core, is likely to undergo a similar fate. My basis for this claim will be that metaphysical reasoning always relies on the Principle of Sufficient Reason, and that the true meaning of this Principle is to be found in the "Theory of Ambiguity" and in the associated mathematical "Theory of Groups."

If I were a Leibnizian mystic, believing in his "preestablished harmony," and the "best possible world" so satirized by Voltaire in "Candide," I would say that the metaphysical importance of the Principle of Sufficient Reason and the cognate Theory of Groups arises from the fact that God thinks multi-dimensionally[3] whereas men can only think in linear syllogistic series, and the Theory of Groups is the appropriate instrument of thought to remedy our deficiency in this respect.

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[2] As far as I am aware, only Scholastic Philosophy has fully recognized and exploited this principle as one of basic importance for philosophic thought.

[3] That is, uses multi-dimensional symbols beyond our grasp.

The founder of the Theory of Groups was the mathematician Evariste Galois. At the end of a long letter written in 1832 on the eve of a fatal duel, to his friend Auguste Chevalier, the youthful Galois said in summarizing his mathematical work,[4] "You know, my dear Auguste, that these subjects are not the only ones which I have explored. My chief meditations for a considerable time have been directed towards the application to transcendental Analysis of the theory of ambiguity. . . . But I have not the time, and my ideas are not yet well developed in this field, which is immense." This passage shows how in Galois's mind the Theory of Groups and the Theory of Ambiguity were interrelated.[5]

Unfortunately later students of the Theory of Groups have all too frequently forgotten that, philosophically speaking, the subject remains neither more nor less than the Theory of Ambiguity. In the limits of this lecture it is only possible to elucidate by an elementary example the idea of a group and of the associated ambiguity.

Consider a uniform square tile which is placed over a marked equal square on a table. Evidently it is then impossible to determine without further inspection which one of four positions the tile occupies. In fact, if we designate its vertices in order by A, B, C, D, and mark the corresponding positions on the table, the four possibilities are for the corners A, B, C, D of the tile to appear respectively in the positions A, B, C, D; B, C, D, A; C, D, A, B; and D, A, B, C. These are obtained respectively from the first position by a null rotation (*I*), by a rotation through 90° (*R*), by a rotation through 180° (*S*), and by a rotation through 270° (*T*). Furthermore the combination of any two of these rotations in succession gives another such rotation. Thus a rotation *R* through 90° followed by a rotation *S* through 180° is equivalent to a single rotation *T* through 270°, i.e., $RS = T$. Consequently, the "group" of four operations *I*, *R*, *S*, *T* has the "multiplication table" shown here:

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[4] My translation.

[5] It is of interest to recall that Leibniz was interested in ambiguity to the extent of using a special notation v (Latin, *vel*) for "or." Thus the ambiguously defined roots 1, 5 of $x^2-6x+5=0$ would be written $x=1 \text{ v } 5$ by him.

	I	*R*	*S*	*T*
I	*I*	*R*	*S*	*T*
R	*R*	*S*	*T*	*I*
S	*S*	*T*	*I*	*R*
T	*T*	*I*	*R*	*S*

This table fully characterizes the group, and shows the exact nature of the underlying ambiguity of position.

More generally, any collection of operations such that the resultant of any two performed in succession is one of them, while there is always some operation which undoes what any operation does, forms a "group."

In the discussion of our main topic we turn first to some illustrations of the Principle of Sufficient Reason in the mathematical and physical domains. In the biological and psychological domains there appears to be as yet no scope for the Principle, due perhaps to the fact that our knowledge is not yet deep enough in these more complicated fields; however, it will be seen that in a certain metaphysical sense, the Principle is already of speculative importance for biological thought. In the social domain it will be possible to point out how the Principle plays a decisive rôle in certain special problems; but here our knowledge is still less. Finally, I shall attempt a general quasi-mathematical formulation of the Principle itself, and venture a heuristic conjecture as to its ultimate rôle in scientific and philosophic thought.

Let us begin with some simple remarks concerning the nature of the ordinary complex number-system which, of course, is the most important technical weapon of mathematics. In the first place, we may note that there is a kind of similarity between the processes of addition and of multiplication, which is not so complete as to give rise to essential ambiguity, but yet is extensive enough to permit of the invention of logarithms replacing multiplication by addition. Secondly, there is a thorough-going similarity between the relations of "greater than" ($>$) and "less than" ($<$), holding between pairs of numbers. But, more significantly still, the number-system is finally regarded as completed by the inclusion of the "imaginary unit," $i=\sqrt{-1}$; and this unit is essentially ambiguous in that one might equally well write $i=-\sqrt{-1}$. In either case we have the basic equation $i^2=-1$. Here the underlying group has only two operations, A, B, respectively taking i into itself and into its negative, with the multiplication table:

	A	B
A	A	B
B	B	A

This ambiguity, inherent in our number system, affords a very simple and mathematically unexceptionable application of the Principle of Sufficient Reason in the following way.

Suppose that there be given an algebraic equation with real coefficients (as for instance $x^2-2x+2=0$), with one imaginary root, $a+bi$ ($1+i$ for the quadratic equation just mentioned). Then *without further argument*, on the basis of our Principle alone, it follows that $a-bi$ ($1-i$ in the special case) is also a root of the equation. For why should $a+bi$ be a root any more than $a-bi$? The usual proof that $a-bi$ is also a root, involves the property of a complex number $a+bi$, that, if it vanishes, both the real coefficient a and the imaginary coefficient b vanish separately.

This example is so illuminating that it is worth while to state the underlying argument based on the Principle a little more fully:

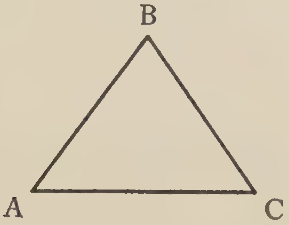
(1) The ordinary complex number system is regarded as given, and contains the imaginary unit i; and it is observed that if i be replaced by $-i$ the set of underlying postulates still holds, with the modified imaginary unit $-i$;

(2) It is given that $x=a+bi$ is a root of an algebraic equation with real coefficients $f(x)=0$, i.e., $f(a+bi)=0$;

(3) Changing i to $-i$ throughout, we infer without further argument that $f(a-bi)=0$, in other words that $a-bi$ is also a root. It is to be noted that, since the coefficients of the equation are *real*, the equation itself is not altered when i is replaced by $-i$.

There is also a kindred suggestive general algebraic principle, that when numbers are algebraically defined in an unambiguous manner, they are *rationally* known, i.e., are given by ordinary fractions. It was by means of the extension of this last principle to ambiguously-defined algebraic quantities (e.g., as $\sqrt{2}$, ambiguously defined by $x^2=2$), and the related theory of Permutation Groups that Galois was able to formulate a definitive theory of the solution of algebraic equations by radicals.

A more important use of the Principle of Sufficient Reason is afforded by elementary Euclidean geometry. Here the "group of motions" is the one which is important. The operations of this group move geometrical figures in space without changing their dimensions, and thus allow us to determine the congruence of figures by superposition. As a very simple illustration, consider an isosceles triangle ABC with given $\angle ABC$, and given equal sides $BA=BC$. Clearly the Principle of Sufficient Reason indicates that the base angle, $\angle BAC$, is equal to the other base angle, $\angle BCA$. For why should it be greater rather than less? In fact, if we turn the triangle over so that BC takes the position BA, then BA must fall along the line BC, while the point A must coincide with C. But, philosophically speaking, the gist of the argument depends on the Principle of Sufficient Reason rather than upon a use of superposition.



Now there is inherent in geometry a more extensive ambiguity than that of *congruent* figures, namely that of *similar* figures. According to Couturat[6] the corresponding larger group of similitudes is really the one appropriate to an approach to geometry from the standpoint of intuition (Sufficient Reason).

In Leibniz's own attempt to construct a "geometric calculus" of position, he makes various uses of the group of similitudes, but finally abandons it in favor of the Euclidean group of motions. Couturat believes that this was the basic reason for Leibniz's lack of success in his geometric calculus. Nevertheless Leibniz's attempt was of the highest interest: for Boole in his *Algebra of Logic* and Grassmann in his *Calculus of Extension* "later justified the most daring conceptions of Leibniz, in showing that these were not dreams, but prophetic intuitions which anticipated by nearly two centuries the progress of science and the human spirit."[7]

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[6] L. Couturat, *La Logique de Leibniz* (Paris, 1901), pp. 410–416.

Possibly a simple system of postulates of my own for geometry[8] is nearest in spirit to the suggestion of Couturat's, for my approach is based on the group of translations (scale), the group of rotations (protractor), and the group of expansions of figures about a fixed point (pantograph), and these are the three principal "subgroups" making up the group of similitudes in the plane.

Indeed, in any mathematical approach to geometry, it will be found that the choice of postulates is largely determined by considerations based on an underlying group, whether that of motions as in Euclid or of similitudes as suggested by Couturat, or of some other group like the so-called projective group.

It may be remarked in passing that for the primitive mind unacquainted with formal geometrical truth, the form of the circle, as embodied in the full moon, for instance, must have symbolized somehow the essential characteristics of the group of rotations about the center of the circle and of the associated ambiguity of all directions through the center.

It is interesting that Leibniz himself had the erroneous belief that, with a more thoroughgoing use of his Principles of Contradiction and of Sufficient Reason, it would be possible to prove the familiar axioms of Euclid. Thus he says: "Far from approving the acceptance of doubtful principles, I would have people seek even the demonstration of the axioms of Euclid. . . . And when I am asked the means of knowing and examining innate principles I reply . . . that . . . we must try to reduce them to first principles, i.e., to axioms which are identical or immediate by means of definitions which are nothing but a distinct exposition of ideas."

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[7] My translation of Couturat.

[8] *Annals of Mathematics*, vol. 33 (1932), pp. 329–345.

Another very important mathematical subject in which the Principle of Sufficient Reason enters is that of the Theory of Probability. If a coin is tossed, it is regarded as equally probable that the coin fall heads or tails in the successive throws. For how can the irrelevant marking on the two sides of the coin make either side more likely than the other? It is clear that the weightiest practical decisions must generally be guided by similar estimates of probability, whether these be in the successful conduct of the affairs of a nation, a business house, or an individual.

It is, however, in physics that one finds still more remarkable instances of the application of the Principle of Sufficient Reason. Only a few of these can be sketched here, of course.

We will begin with the case of two "equal forces" F acting upon a point P.[9] If we admit that these are equivalent to a single resultant force R, then it is obvious from our Principle that the resultant must lie in the plane of the two forces. For why should it lie upon one side of this plane rather than upon the other? Here the tacit assumption is that the associated group is once more the group of motions characteristic of ordinary geometry. It is clear, for the same reason, that this resultant must fall along the internal bisector of the lines of the two forces (see the figure).

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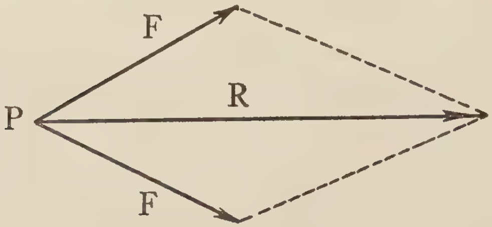
[9] A similar application is to a balance with equal weights in the two pans of the balance.



Furthermore, by a more elaborate but valid use of the Principle of Sufficient Reason, we may prove that in general the resultant of any two forces *must* be that given by the familiar Parallelogram Law[10]. This is the type of "proof" used for instance by the French astronomer and mathematician Laplace in his *Mécanique Céleste.*

Much of classical mechanics may be approached in a similar spirit. For example, the attracting gravitational force acting between two point masses must evidently lie along the line joining the two masses, by the same Principle; and the forces of action and reaction must be equal – for why should one force be greater than the other?—and there can be no absolute unit of length—for why is one unit to be preferred rather than another? Thus it follows from the Principle that the attractive force varies as some power of the distance.

If now we further impose the (philosophically reasonable) requirement that the mutual interaction is small at large distances apart, we find that the force must vary as some negative power of the distance. Thus we are led practically to the inverse first or second powers by use of the Principle of Sufficient Reason alone; and the planetary laws of Kepler indicate at once that the inverse second power of Newton has to be chosen. Thus from the standpoint of the Principle of Sufficient Reason, based on the ambiguity of the group of motions and similitudes, the gravitational law of Newton appears as *almost* inevitable.

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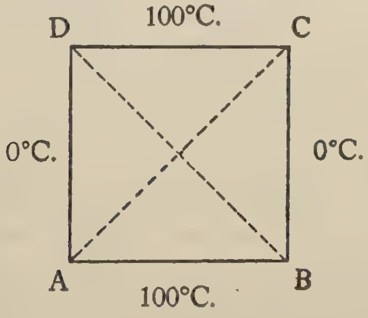
[10] See the Note appended to this lecture.

It is worthy of remark in this connection that Leibniz, searching for the invariable "substances" underlying the changing appearance of mechanical systems such as the solar system, was led to formulate the laws of the conservation of momentum, and of energy or *vis viva*, so important for modern physics.

Another instructive illustration of the power of the Principle of Sufficient Reason in certain physical questions is furnished by the following special problem. Suppose that a uniform square sheet of metal has two opposite sides maintained at the freezing point of water (0°C.) while the other two sides are maintained at the boiling point of water (100°C.). Suppose further that no heat is conducted away except along the edges, and that the *additivity* of states of temperature is assumed. It is asked, on the basis of these very scanty assumptions: what are the permanent temperatures set up along the two diagonals?



The Principle of Sufficient Reason alone suffices to show that along these two lines the temperature must be everywhere exactly 50°C. In fact, rotate the square by half a revolution about the diagonal *AC* (see the above figure), so that pairs of adjacent sides are interchanged while the individual points of the diagonal *AC* remain fixed. By the additivity assumed, the resultant combined temperature will be 100° along all of the sides and double the temperature along the fixed diagonal. But if the temperature is permanently 100°C. along the boundary, it will be 100°C. everywhere inside, so that the temperature all along the diagonal *AC must* have been exactly 50°C.; and the same must be true along the other diagonal *BD*.

Evidently this is a remarkable conclusion, made on the basis of limited assumptions without any calculation! This is possible because of the peculiar symmetry of the problem which permits the use of the underlying group of rotations of the square in order to arrive at a precise quantitative result.

These illustrations are sufficient to show the exceedingly great suggestiveness of the Principle of Sufficient Reason for physics. It has been used, more or less tacitly, in almost all classical physics of the Newtonian type.

However, in the subsequent electromagnetic era inaugurated by Faraday and Maxwell, it seemed until the beginning of the present century as though the Principle of Sufficient Reason had no important part to play. In fact, the ordinary group of motions in no way suggested the peculiar laws interrelating the electric and magnetic lines of force visualized by Faraday. Here one was apparently confronted by mysterious entities of non-mechanical nature which were entirely foreign to ordinary physical intuition.

It is therefore a striking fact that in 1908 the mathematician Minkowski found a more natural way of looking at the phenomena of electromagnetism. He discovered that by introducing as basic a certain "Lorentz" group instead of the group of motions, the observed laws lost their apparent artificiality and took on a character of inevitability from a higher, group-theoretic point of view. Minkowski foresaw clearly the importance of this forward step, made shortly before his death.

The Lorentz group is applicable to a (3+1)-dimensional space-time, and is made up of operations which mix up the three space coördinates and the single time coördinate of two systems in relative motion. At low velocities compared to that of light, the quantitative difference between the group of Euclidean motions[11] and the Lorentz group is insignificant.

The point of view of Minkowski is consonant with that of the special theory of relativity of Einstein (1905) and has been universally adopted in dealing with electromagnetism. It permits us to write in a quite legitimate way the mystical equation

$$186{,}300 \text{ miles} = \sqrt{-1} \text{ seconds.}$$

It also enables us to answer in similar fashion the question: Why is space three-dimensional, and why is time one-dimensional? The answer is that a theory of electromagnetic type can be proved to be possible only in the case of a three-dimensional space and a one-dimensional time!

Here it is interesting to recall that Leibniz, as a systematic philosopher, considered that the fact that space was three-dimensional followed as a geometric necessity from the usual axioms of geometry, for he says: "The triple number of dimensions is determined, not by the reason of the best [i.e., not by a special application of the Principle of Sufficient Reason], but by a geometrical necessity: it is because geometers have been able to show that there are only three mutually perpendicular straight lines which can intersect in the same point." Here of course Leibniz was entirely wrong, since n-dimensional Euclidean geometry is just as possible as three-dimensional. However, if he were living today, he would undoubtedly "explain" the triple dimensionality of space on the electromagnetic basis outlined above.

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[11] Together with a change in origin of time.

A somewhat analogous situation arises for number systems of so-called non-commutative type, different from ordinary numbers and yet permitting of the operation of division: here the only type is essentially that of the four-dimensional quaternions discovered by the Irish mathematician William Rowan Hamilton.

It was Minkowski's paper of 1908 which stimulated Einstein to formulate his general gravitational theory of 1915.In this theory, too, the Principle of Sufficient Reason played a notable part, for it was the complete directional ambiguity of space about the central Sun which alone enabled Einstein to draw his conclusions. He did not explicitly appeal to the Principle, but it was used just the same. As a matter of fact, the gravitational theory of relativity has so far only been successfully applied in this very special, highly symmetrical case of a large spherical mass.

In passing it may be observed that Einstein's general theory of relativity in its first form can itself effectively be criticized on the basis of the same Principle of Sufficient Reason! In fact, if there were only a single central body, this body might be rotating *according to that theory*. But with respect to what could it rotate?

It is remarkable that Leibniz himself should have been led to take a specifically relativistic point of view towards space and time. No doubt this suited him because it stood in opposition to the absolutistic point of view of his great rival Newton. But Leibniz was led to adopt this position primarily through the Principle of Sufficient Reason, as the following quotation shows: "I hold space, and also time, to be something purely relative. Space is an *order of coexistences* as time is an *order of successions.* Space denotes in terms of possibility an order of things which, in so far as they exist together, exist at the same time, whatever be their several ways of existing." Again he says, "There are many ways of refuting the imagination of those who take space to be a substance, or at least something absolute. I say that, if space were an absolute Being, it would be impossible to give a sufficient reason for anything that might happen, yet this principle is with us an axiom." He then goes on to give a proof of his assertion, based on the indistinguishability of different points of ideal space and of different instants of ideal time, which makes it inconceivable that Deity selects any one point or instant in preference to others. Here evidently it is the ambiguity of space corresponding to the group of motions, and the ambiguity of time corresponding to the group of time measurements, that brings in the Principle.

Thus once more a great philosopher has revealed a prophetic insight; and it may be added that the French philosopher Bergson was led on purely philosophical grounds to formulate the (relativistic) idea of "local time" even before Lorentz and Einstein were led to do so on the basis of the celebrated Michelson-Morley experiment and the equations of Maxwell.

In the latest quantum-mechanical phase of physics, the attempt to maintain a conceptual grasp of physical law has been at least temporarily abandoned in favor of a purely formal attack by means of mathematical guesswork. This process, it must be admitted, has so far been strikingly successful. Nevertheless I believe the day not to be far distant when less artificial methods must again be employed.

If before Newton's time the laws of Kepler for the motion of two bodies had been used as a basis for similar shrewd conjectures concerning the three-body problem presented by the Sun, Earth, and Moon, the proper formulas would probably have been obtained; for there were always the known observational facts to guide one *ad hoc* when in difficulty. It is such an *ad hoc* "explanation" that quantum-mechanics seems to me to provide for the treatment of spectroscopic and allied phenomena. Just as Newton's law of gravitation, of conceptual type and largely based on the Principle of Sufficient Reason, would have displaced such an artificial treatment of the three-body problem, so we may expect an alternative conceptual approach to quantum-mechanical laws to be found and adopted.

In the biological and psychological domains there is as yet little or no occasion to use the Principle of Sufficient Reason, except in the extremely vague way which we proceed to indicate.

Physicists have always treated matter as mere dead stuff, although, from the philosophic point of view, matter must be thought of as the potential abode of life. This tacit hypothesis of the physicists is probably a useful one in our present stage of physical knowledge. But an unfortunate effect of it has been to make matter seem alien to life, although recently there are signs that the attitude of biologists is changing in this respect.

However, Leibniz avoided this error of thought, for his theory of monads led him, at least in the early stages of his philosophical development, to locate souls in points, death itself being merely the contraction of a monad dominating an "entelechy" to a single point. The Leibnizian employment of the Aristotelian concept of entelechy in order to explain the activity of living organisms may be regarded as an important biological advance. Such an entelechy was based on the concept of a dominant monad in a monadic system. Leibniz envisaged the existence of continuous series of such monadic beings. These may be illustrated not only by cells and other organisms, but by insect communities, and by other forms as yet unknown. Leibniz said in this connection "I definitely avow that there are in the world animals as much larger than ourselves, as we are larger than microscopic animalcules." One is reminded in this connection of the incident related by his commentator Hantschius, that Leibniz once remarked to him over a cup of coffee, "There may be in this cup a monad which will one day be a rational soul!"

Of course the existence of monads was "deduced" from his fundamental Principles of Contradiction and Sufficient Reason, but it is difficult to see the cogency of Leibniz' vitalistic arguments.

As far as this ever-recurring conflict between mechanistic and vitalistic points of view among biologists is concerned, it seems to me that it may be at once decided in favor of the vitalists by a reasonable philosophic use of the Principle of Sufficient Reason: *for, no sufficient reason for the observed behavior of living organisms is to be found in purely mechanical systems.* For example, from a purely mathematical point of view a system of Newtonian type would necessarily exhibit a type of eternal Nietzschean recurrence, foreign to living things!

In the fivefold hierarchy of the levels of knowledge—mathematical, physical, biological, psychological, social—the social level is of course the most complex and difficult of all. Nevertheless it is already clear that the Principle of Sufficient Reason is likely to play a definite rôle in certain technical problems.

My first illustration is taken from the question of the proper apportionment among the several states of the assigned total of Representatives in Congress (referred to in the preceding lecture). Huntington has emphasized the fact that the prevailing Willcox system of apportionment is based on the individual share, n/N (N = population, n = number of Representatives), whereas there is an ignored but symmetric system based on the idea of the Congressional district N/n. Huntington's own system is of an intermediate type, favoring neither the Congressional district nor individual share. It seems obvious then that, from the point of view of our Principle as it would have been interpreted by Leibniz, the Huntington system is to be logically preferred to either of the other two systems.

My second illustration will be taken from the field of the law, and I am much indebted to my colleague Roscoe Pound for supplying me with it. The case presented is of the following general character. Two small groups of individuals, A and B, request that electric light facilities be extended from the town to their respective nearby communities, the cost of this extension to be divided between them. How should the cost be allocated? As I understand it, the accepted method would be as follows: All of the interests involved and reasonable ways of distributing the cost would be considered, and then a compromise would be struck between these. For example, the basis might be the number of individuals served, or the amount of electricity to be used, or the costs of installation separately, etc. All these, in so far as really different and equally valid, would receive the same consideration in the final decision. It is assumed of course that the case has not been already disposed of by earlier precedents. Evidently this is the proper method of solution on the basis of our Principle.

These two examples suggest that, in the systematic treatment of ethical questions, the Principle of Sufficient Reason may sometimes afford the best way out of an otherwise insoluble problem.

In conclusion I would like to mention one major philosophic result to which the Principle of Sufficient Reason seems to point. It is closely allied with Leibniz' own conclusion that this is the best of all possible worlds, but in reality quite distinct from it. Leibniz elaborates this conclusion in his statement: "It follows from the supreme perfection of God that in producing the universe he has chosen the best possible plan, the greatest variety combined with the most perfect order; ground, place, times arranged as well as possible; the maximum effect received by the simplest means; as much power, knowledge, well-being and goodness in creatures as the universe would admit. . . . Otherwise, it would be impossible to find a reason why things are thus rather than otherwise." In Leibniz's feeling that the root-notions will be found to be few in number, he comes still closer to the idea which I have in mind: says Leibniz, "I believe there are only a few primitive decrees which regulate the consequences of things." Such principles as that of "least action" in dynamics, discovered independently by Leibniz and Maupertuis, seemed to Leibniz to be corroborative of this mystical truth.

Another aspect of the same idea was tacitly employed by Newton when he selected the second power of the distance in formulating his gravitational law, rather than some very nearby power. It may be remarked that such quasi-aesthetic feelings constantly guide the efforts of any successful system-builder in science or philosophy.

Einstein, too, has expressed a variation of the same theme n terms which are perhaps intended to be theological only in an allegorical sense. He declares: "*Raffiniert is der Herr Gott, boshaft ist Er nicht*"—God is subtle, He is not malicious.

Very recently an analogous thesis has been advanced by Sir Arthur Eddington, seeming at first sight to deny the necessity of experimentation. Says Eddington: "Unless the structure of the nucleus has a surprise in store for us, the conclusion seems plain—there is nothing in the whole system of laws of physics that cannot be deduced unambiguously from epistemological considerations. An intelligence, unacquainted with our universe but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment. . . . For example, he would infer the existence and properties of radium, but not the dimensions of the earth."

In making this statement, Eddington apparently overlooks the fact that today we are surrounded by instruments of precision; in other words all of us live in a veritable scientific laboratory, and thus acquire easily an understanding of certain facts of nature which the keenest observer could not have guessed at in ancient times. Eddington regards theories thus arrived at as essentially subjective, but I do not see how this alters the basic significance of his mystical conclusion.

All of these somewhat different points of view indicate a prevailing faith that the Order of Nature is supernal, combining logical simplicity and inevitability in a most remarkable way. Ordinary geometry affords such an account of space; Newtonian dynamics, of the behavior of matter in motion; electromagnetism, of the facts of light and electricity; etc. This belief is grounded, on the one hand, in scientific experience and, on the other, in the successful use of the Principle of Sufficient Reason in more or less metaphysical form.

Up to the present point my aim has been to consider a variety of applications of the Principle of Sufficient Reason, without attempting any precise formulation of the Principle itself. With these applications in mind I will venture to formulate the Principle and a related Heuristic Conjecture in quasi-mathematical form as follows:

PRINCIPLE OF SUFFICIENT REASON. If there appears in any theory T a set of ambiguously determined (i.e. symmetrically entering) variables, then these variables can themselves be determined only to the extent allowed by the corresponding group G. Consequently any problem concerning these variables which has a uniquely determined solution, must itself be formulated so as to be unchanged by the operations of the group G (i.e. must involve the variables symmetrically).

HEURISTIC CONJECTURE. The final form of any scientific theory T is: (1) *based on a few simple postulates; and* (2) *contains an extensive ambiguity, associated symmetry, and underlying group G, in such wise that, if the language and laws of the theory of groups be taken for granted, the whole theory T appears as nearly self-evident in virtue of the above Principle.*

The Principle of Sufficient Reason and the Heuristic Conjecture, as just formulated, have the advantage of not involving excessively subjective ideas, while at the same time retaining the essential kernel of the matter.

In my opinion it is essentially this principle and this conjecture which are destined always to operate as the basic criteria for the scientist in extending our knowledge and understanding of the world.

It is also my belief that, in so far as there is anything definite in the realm of Metaphysics, it will consist in further applications of the same general type. This general conclusion may be given the following suggestive symbolic form:

$$\text{Metaphysics} \longleftrightarrow \begin{array}{c}\text{Principle of}\\ \text{Sufficient Reason}\end{array} \longleftrightarrow \begin{array}{c}\text{Theory of Ambiguity}\\ \text{and Groups.}\end{array}$$

While the skillful metaphysical use of the Principle must always be regarded as of dubious logical status, nevertheless I believe it will remain the most important weapon of the philosopher.

NOTE ON THE LAW OF THE PARALLELOGRAM OF FORCES

In order to show how this law may be regarded as a kind of consequence of the Principle of Sufficient Reason, we begin by laying down the following three postulates:

(I) Collinear vector forces at a point O have a resultant force in the same line whose magnitude is the (algebraic) sum of the magnitudes of the two constituent forces.

(II) Any two forces have a unique resultant. The combination of forces into resultants is associative and commutative:

$$(F+G)+H=F+(G+H);$$
$$F+G=G+F.$$

The resultant of forces varies continuously with the constituent forces.

(III) *Principle of Sufficient Reason.* The resultant of two forces is independent of the choice of axes of reference and of the unit of force.

It is easy to show on the basis of these postulates that the law of composition of forces is simply the law of vector addition, embodied in the so-called Parallelogram Law.

To do so we remark first that the resultant H of two non-collinear forces F and G cannot lie in the line of either of these forces. For from

$$H=F+G$$

ve obtain by (II) and (I)

$$(-F)+H=((-F)+F)+G=G.$$

Hence if H and $(-F)$ are collinear, G will be collinear with hem by I, which is impossible. Thus H and F (and similarly H and G) cannot be collinear.

We also observe that the unique resultant H of two forces F and G (see II) must lie in their plane by the Principle of Sufficient Reason (III); for why should this resultant lie on one ide of this plane rather than the other?

Now let us select two orthogonal axes in any plane through O and consider forces X and Y along these axes in the chosen ositive sense, with magnitudes $\cos \alpha$ and $\sin \alpha$ respectively here α ranges from 0 to $\pi/2$. For $\alpha=0$ the resultant is a nit vector along the x-axis; for $\alpha=1$ the resultant is a unit ector along the y axis. As α varies from 0 to $\pi/2$ the resultnt vector OP varies continuously by (I). Thus P describes continuous curve (see figure 1).

It is clear that this curve can not intersect the x or y axis, or $0<\alpha<\pi/2$, by our first remark, and so must lie wholly ithin the first quadrant as indicated in the figure. Conseuently OP must take any desired direction in the quadrant t least once, and, since the resultant by the same Principle independent of the unit selected we may decompose any rce H whose direction falls in this quadrant into two such rces X and Y,

$$H=X+Y.$$

But there cannot be more than one such decomposition, r if

$$X+Y=X'+Y',$$

en

$$(X-X')+(Y-Y')=0;$$

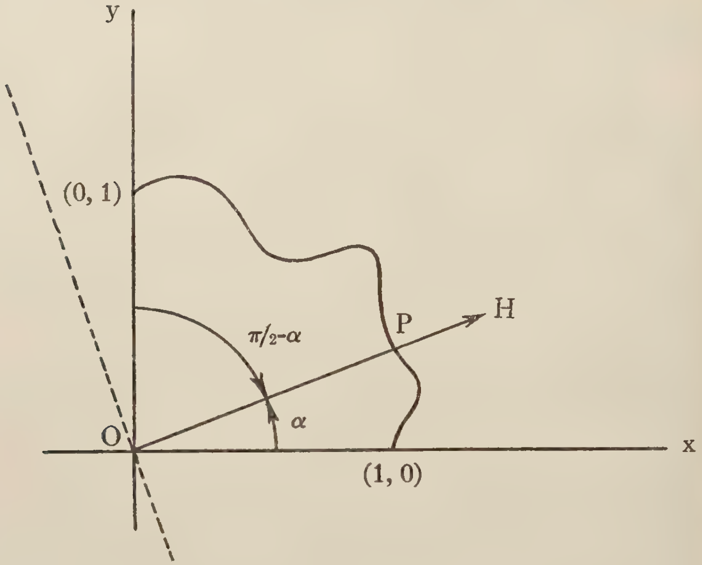

Fig. 1.

whence by our first remark

$$X = X', \quad Y = Y'.$$

Furthermore we see by the intrinsic geometry of the same figure, based upon our Principle, that

$$X = h\varphi(\alpha) \quad Y = h\varphi(\pi/2 - \alpha) \tag{1}$$

where h and α designate the magnitude and angle of H, and φ is a continuous function of α such that

$$\varphi(0) = 1,\ \varphi(1) = 0, \qquad 0 < \varphi(\alpha) \text{ for } 0 < \alpha < \pi/2.$$

Here we have let X and Y stand (ambiguously) for the magnitudes of the vectors X and Y of specified directions.

Next let us resolve X and Y along the direction of H and the perpendicular direction (see the same figure). Clearly we obtain

$$\begin{cases} h = X\varphi(\alpha) + Y\varphi(\pi/2-\alpha) \\ 0 = -X\varphi(\pi/2-\alpha) + Y\varphi(\alpha). \end{cases}$$

The second of these relations reduces to an identity in virtue of (1) but the first yields by combination with (1)

$$\varphi^2(\alpha) + \varphi^2(\pi/2-\alpha) = 1, \tag{2}$$

and so in particular $\varphi(\pi/4) = 1/\sqrt{2}$.

This shows that we have always

$$h^2 = X^2 + Y^2 \tag{3}$$

n accordance with the Parallelogram Law.

To complete our deduction of that Law we now consider two forces of equal magnitude h making an angle $\alpha (0 < \alpha < \pi/2)$ with one another as in figure 2.

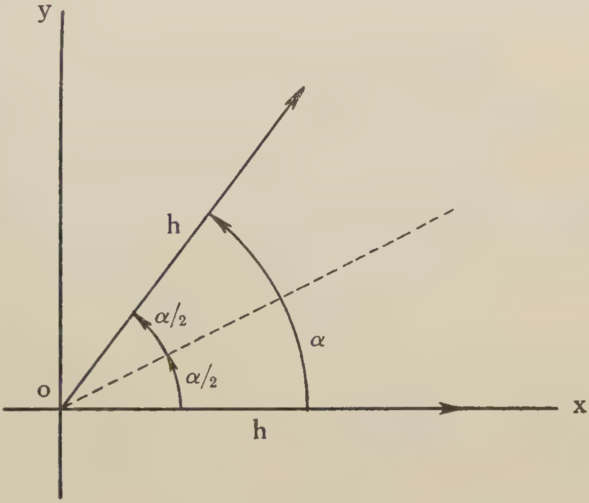

FIG. 2.

We first resolve both of these forces along the line of one them and in a perpendicular direction obtaining compo- ents.

$$h + h\varphi(\alpha) \quad \text{and} \quad h\varphi(\pi/2-\alpha)$$

respectively. By (3) and (2) the resultant is therefore of magnitude

$$\{[h+h\varphi(\alpha)]^2+h^2\varphi^2(\pi/2-\alpha)\}^{\frac{1}{2}}=h[2(1+\varphi(\alpha))]^{\frac{1}{2}}.$$

Secondly we resolve along the bisector of the lines of the forces and perpendicularly to this bisector, obtaining components

$$2h\varphi(\alpha/2) \quad \text{and} \quad 0$$

respectively. Hence we conclude that

$$\varphi^2(\alpha/2)=\frac{1+\varphi(\alpha)}{2}, \qquad 0<\alpha<\pi/2. \tag{4}$$

From the companion formulas (2) and (4) we conclude that for all α of the form $m\pi/2^n$ we have $\varphi(\alpha)=\cos\alpha$. In fact we already have found that

$$\varphi(0)=1,\ \varphi(\pi/2)=0,\ \varphi\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}};$$

and by bisection we find next

$$\varphi\left(\frac{\pi}{8}\right)=\cos\pi/8,\ \varphi\left(\frac{3\pi}{8}\right)=\cos\frac{3\pi}{8};$$

and so forth. But $\varphi(\alpha)$ is continuous and so we see that for all α we have $\varphi(\alpha)=\cos\alpha$. Thus (1) takes the form

$$X=h\cos\alpha,\ Y=h\sin\alpha \tag{5}$$

in accordance with the Parallelogram Law of resolution along these axes. Furthermore it is clear that the formula (5) holds for any α whatsoever. Hence it follows that any two forces H_1 and H_2 combine in the desired manner since we have

$$X_1=h_1\cos\alpha_1,\ Y_1=h_1\sin\alpha_1;\ X_2=h_2\cos\alpha_2,\ Y_2=h_2\sin\alpha_2,$$

with resultant components,

$$h_1\cos\alpha_1+h_2\cos\alpha_2,\ h_1\sin\alpha_1+h_2\sin\alpha_2,$$

in the selected directions.

III

RECTILINEAR DRAWING[1]

I. INTRODUCTION

SUPPOSE that one draws by means of a well-sharpened pencil many straight lines of uniform, narrow breadth extending across a sheet of white paper. What kinds of drawings may be made with this extremely restricted and artificial medium? Here we must hold the pencil point against the paper with a single definite pressure, while moving the point always at the same velocity.

More precisely, in the idealized form of the problem here considered, we take these straight lines to be of microscopic width, so as not to be individually discernible, and to be very numerous. The problem is then to determine whether or not a given (idealized) wash drawing can be reproduced by such (idealized) means, and further, just how it is arrived at.

Let us begin with a very simple illustration of such a problem: The given drawing is to be such that the surface density of lead deposited—determining the degree of blackness—is inversely proportional to the distance from a fixed point O. Can this particular drawing be made by such rectilinear means?

The answer is clearly affirmative. For imagine a very large number of lines drawn through O in an equiangular distribution (see fig. 1).[2] The amount of lead Q deposited within a circle of radius r having O as center is evidently proportional to r: $Q=kr$. Hence the amount of lead dQ in the ring between a circle of radius r and one of radius $r+dr$ (dr, an "infinitesimal") is $dQ=kdr$. But the area dA of this ring is $2\pi rdr$, since $A=\pi r^2$ is the area of a circle of radius r. Consequently the surface density of the lead is given by the ratio $dQ/dA=k/2\pi r$. Hence the density obtained is inversely proportional to the distance r from the fixed point O, as required.

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[1] In a paper "On drawings composed of uniform straight lines" which has just appeared in Liouville's *Journal de mathématiques pures et appliqueés,* I have presented the same topic from a strictly mathematical point of view.

There arises similarly in any such problem a fundamental density function F, depending upon position in the plane and corresponding to the degree of blackness of the drawing which ranges from white through gray to black. The extreme cases $F=0$ and $F=\infty$ correspond to white and black respectively. We may think of F as measured by the depth of the deposit of lead on the paper. Of course in actual practice not only has F a certain effective maximum, after which the lead does not adhere to the paper, but F will change gradually from point to point. However in our idealization of the problem we shall not always require F to be finite and continuous. Evidently in the special case considered above F becomes infinite at the point O.

On the other hand we shall always assume that the amount of lead laid down, $\int FdA$ (dA, element of area) is finite in any finite part of the plane.

An interesting variant of the general problem of rectilinear drawing specified above is obtained when we allow rectilinear erasures to be made after the drawing has been completed, with the natural requirement, of course, that no lines already drawn are to be erased. Here if F_d is the

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[2] The drawings shown in this paper have been very kindly supplied by Mr. David Middleton, to whom I desire here to express my warm appreciation.

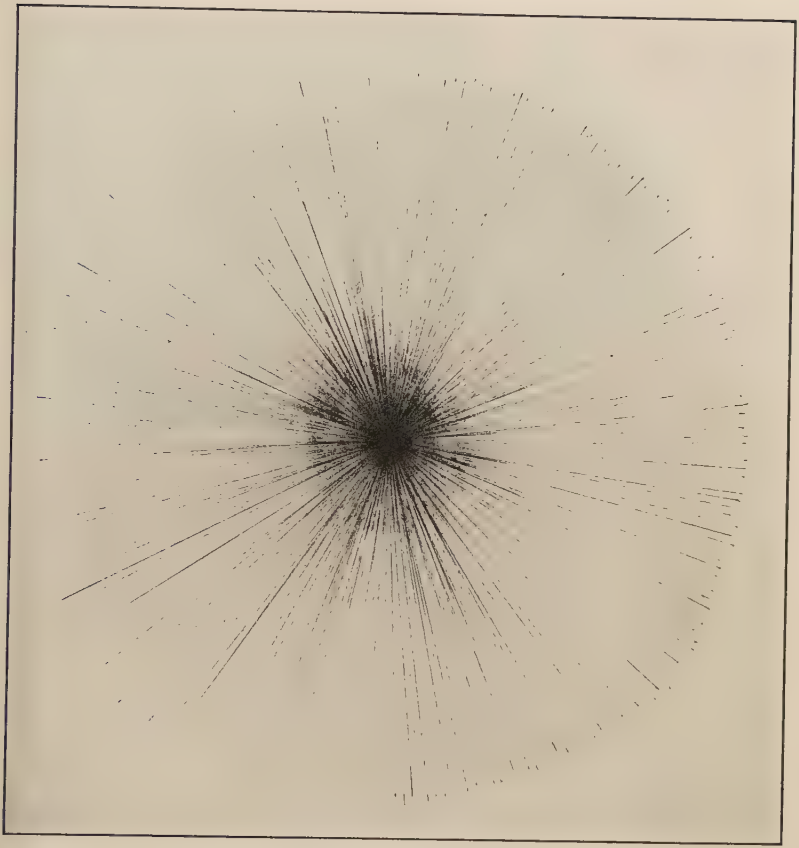


FIG. 1

density for the drawing, and F_e is the similar density corresponding to the erasure (so that $F_d \geqq F_e$ everywhere), then the final drawing clearly corresponds to the density function $F = F_d - F_e \geqq 0$.

A second variant of our problem is obtained if we allow only a single *uniform* erasure all over the plane, i.e., modify $F \geqq k > 0$ to $F - k$, where F is the density function for the given drawing and the constant k corresponds to the uniform erasure in question.

For our later purposes it is convenient to select a certain point O in the plane as center for a system of rectangular coordinates x, y and of related polar coordinates r, θ. Here r is regarded as positive or negative while θ is an angular coordinate of period 2π so that (r, θ) and $(-r, \theta+\pi)$ will

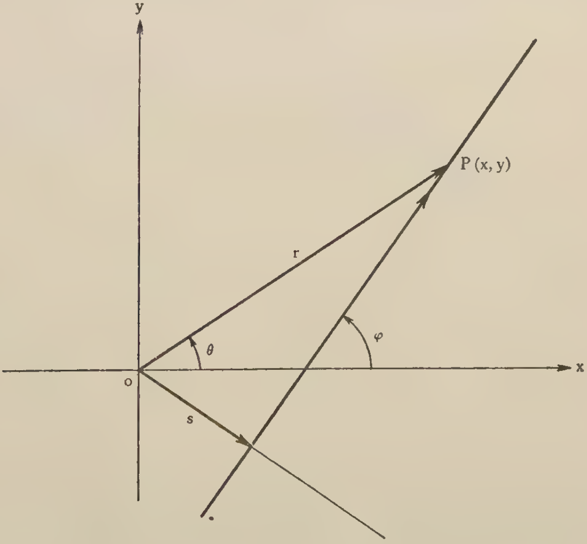


FIG. 2

denote the same point. Thus we may regard the density F as a function of r and θ of period 2π: $F = F(r, \theta)$, with $F(-r, \theta+\pi) = F(r, \theta)$. Similarly we may specify a line l

in the same plane by the coordinates s and φ as indicated in the figure. These are the usual line coordinates useful for our purposes.

2. ONE-PARAMETER FAMILIES OF STRAIGHT LINES

Let us imagine now that a family of straight lines be drawn which depends on a single parameter. We consider first the following case: the straight lines envelop a regular convex arc C with continuous curvature $1/\rho$, along which the points of tangency of the lines are distributed with an assigned frequency $f(s)$, where s designates arc length along the curve C (see figure 3).

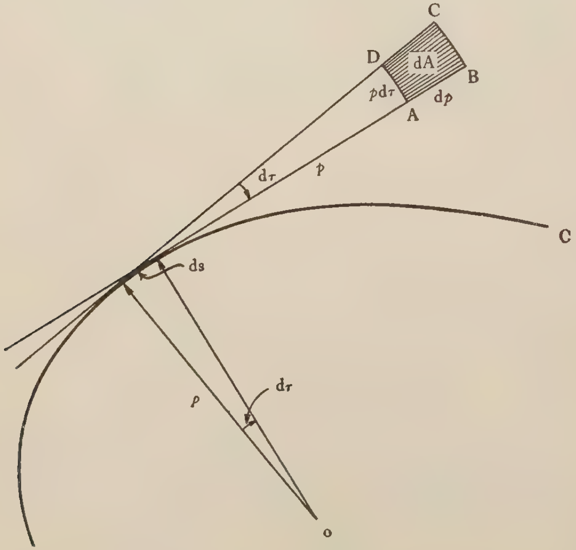


FIG. 3

This means that to get an approximation to the limiting form of drawing we may distribute the tangent lines along the curve C at points $f(s)\ ds$ apart, (ds, a small fixed increment of arc). It is obvious that the function $f(s)$ is a kind of distribution function, specifying the distribution of the given one-parameter family of straight lines.

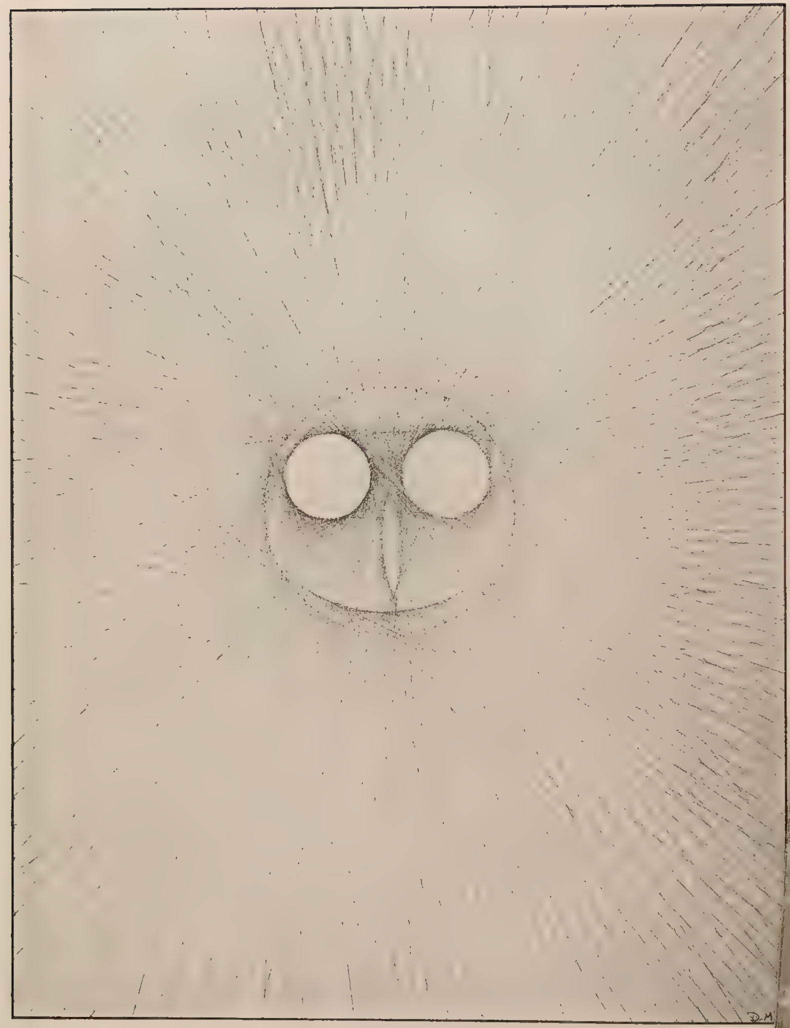


Fig. 4

On considering the above figure it is clear that the area dA of the small curvilinear quadrilateral $ABCD$ is given by $p d\tau dp$. Likewise the amount of lead in $ABCD$ is $f\rho d\tau dp$, since the length of each segment of the straight lines crossing $ABCD$ is dp while the number of these lines is $fds = f\rho d\tau$. Hence the limiting density at a point P is $f\rho/p$: $F = f\rho/p$.

Suppose now that we are given a drawing, and desire to determine whether or not it arises from such a one-parameter family of straight lines. According to what has just been shown, it is clearly necessary for this: (1) that the equation $F = \infty$ defines the convex arc; and (2) that along any tangent to this arc C, F varies inversely as the distance from the point of tangency. If these two conditions are satisfied and if $\rho(>0)$ denotes the radius of curvature at the point of tangency, the desired distribution function $f(s)$ is clearly Fp/ρ.

Before proceeding further it is worth while to note the practical use of such one-parameter families for obtaining outline drawings of curvilinear arcs C. In fact since the density F is infinite along the arc C, this curve appeared as clearly etched. Similarly any set of arcs can be simultaneously etched, although there is no reason to believe that the *shading* of masses will then be as desired. The pumpkin-head drawing of Mr. Middleton herewith shown (figure 4) affords a simple and amusing illustration of the possibilities.

It is interesting to observe that in such an outline drawing the density function F is given as a sum of terms of the form $f(s)\rho/p$ referred to above. For example, in his figure Mr. Middleton has taken $f(s)\rho$ to be the same along every circle and circular arc; this requirement means that the tangents are drawn at equal *angular* intervals along all of the circles and circular arcs, inasmuch as f is inversely proportional to

ρ. Hence in this case the density function at any point is simply given by the sum of the reciprocals of the lengths of tangents which can be drawn from the given point to the circles and circular arcs.

Our second case of a one-parameter family will be the degenerate case in which the curve C enveloped by the straight lines reduces to a single point. For definiteness we will further assume that the frequency of the points of intersection of the radial lines through this point O is given by a continuous function $f(\varphi)$ on the unit circle about the point O; here φ designates the angle which the lines make with a fixed line. It is then apparent that we have $F(r, \theta) = f(\varphi)/r$ as the corresponding density function, and that this density function is continuous everywhere except at the origin $r=0$, where it is obviously discontinuous.

The third and last (one-parameter) case is the completely degenerate case in which there is no envelope and we have a family of parallel straight lines so that $F(r, \theta) = f(r \cos\theta)$ if all the lines make an angle of $\pi/2$ with the initial direction. If we had employed corresponding rectangular coordinates x, y, we would have

$$F(r, \theta) = F^*(x, y) = f(x),$$

or more generally, for an arbitrary direction of the family of parallel straight lines, we would have

$$F^*(x, y) = f(ax+by).$$

Here $f(ax+by)$ is taken as an arbitrary continuous function of its argument $ax+by$.

Now it has long been known that an arbitrary continuous function of x and y can be approximated to as a sum of continuous functions of such a linear variable $ax+by$. Consequently it is obvious that any positive continuous function can be approximated to by drawing suitable parallel families (or parts thereof) and then making suitable erasures of parallel families. More specifically, one need only draw the families of parallel lines appearing in the sum wherever the corresponding f is positive; and then make the necessary erasures wherever f is negative; in fact the two components have then the desired algebraic sum which is positive or zero.

Consequently we conclude that any drawing in the finite part of the plane corresponding to a continuous positive (or zero) density function can be approximated to by means of a finite set of continuous one-parameter families of parallel straight lines and by subsequent one-parameter families of parallel erasures.

3. TWO-PARAMETER FAMILIES OF STRAIGHT LINES

An arbitrary straight line l has been specified by two coordinates s and φ, of which the first is a radial coordinate, the second an angular coordinate of period 2π (see fig. 2); furthermore (s, φ) and $(-s, \varphi+\pi)$ then correspond to the same straight line. There is thus a two-parameter family of such straight lines.

When we attach a continuous two-parameter "distribution function" $f(s, \varphi)$ to such straight lines we mean that the mass due to the straight lines with an angle between φ and $\varphi+d\varphi$ and traversing a small area dA is nearly given by

$$f(s, \varphi)dAd\varphi,$$

where s denotes the distance of some such line through dA from the origin O.

More precisely, consider a series of n angles increasing from 0 to 2π through small equal increments $\Delta\varphi$ so that $n\Delta\varphi=2\pi$; and suppose that, for each such φ, for s increasing from 0 to S by small equal increments Δs so that $n\Delta s=S$, lines of breadth proportional to

$$f(\mu\Delta s, \nu\Delta\varphi)\Delta s\Delta\varphi \quad (\mu, \nu = 1, 2, \cdots n),$$

are drawn (i.e., $f(\mu\Delta s, \nu\Delta\varphi)$ lines of the same breadth). The kind of distribution of lines just specified is essentially independent of both the origin of coordinates and the direction of the initial line. It is clear that in a small area dA the limiting density as n increases without limit will be given by

$$\int_0^{2\pi} f(s, \varphi)d\varphi.$$

Now we have the relationship (see fig. 2) $s = r\sin(\varphi - \theta)$. In consequence we have the following general fundamental equation connecting the density function $F(r, \theta)$ and the distribution function $f(s, \varphi)$:

$$F(r, \theta) = \int_0^{2\pi} f(r\sin(\varphi - \theta), \varphi)d\varphi. \tag{1}$$

Thus, given the continuous distribution function $f(s, \varphi)$, say for $|s| \leqq S$ $(S < +\infty)$ it is immediately possible to determine the corresponding continuous density function $F(r, \theta)$ for $|r| \leqq S$ by the above formula.

Our primary concern is of course the inverse problem: Given a density function $F(r, \theta)$ for $|r| \leqq R < +\infty$, to ascertain whether or not there exists a corresponding distribution function $f(s, \varphi)$ for $|s| \leqq R$, and further to determine all such functions (when they exist). From this point of view the fundamental equation (1) appears as a special type of "linear integral equation of the first kind" for the unknown function $f(s, \varphi)$. Since we are here concerned with determining whether or not a given wash drawing can be reproduced by means of smooth distributions in distance and angle of very many fine indefinitely extended straight lines, the solution $f(s, \varphi)$, as well as $F(r, \theta)$, is in general required

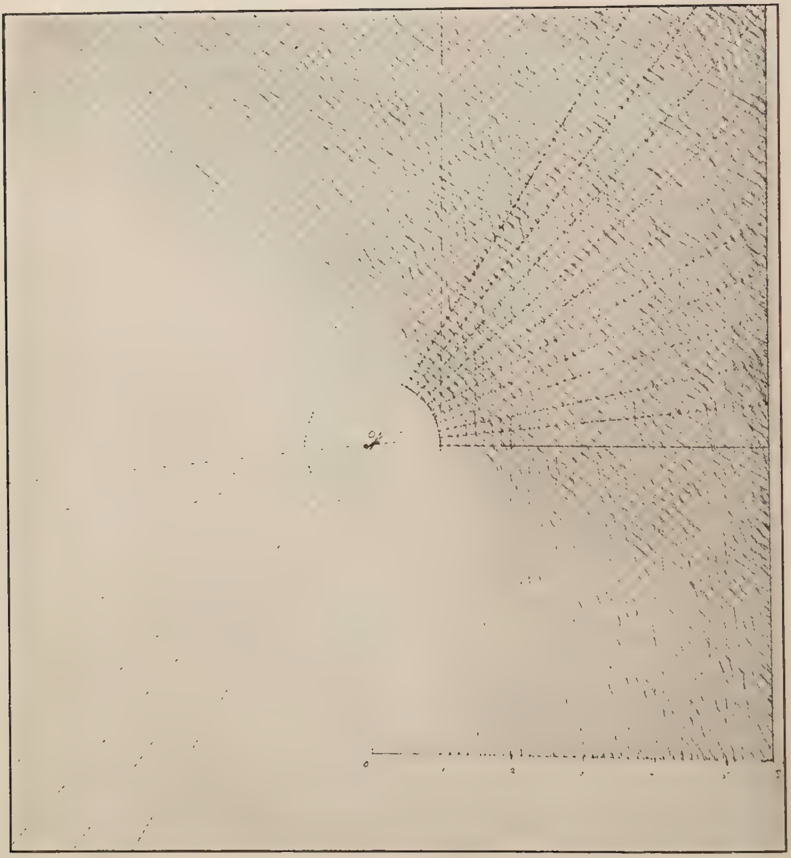


Fig. 5

to be continuous in its arguments. The singular distributions hitherto mentioned are of course not of this type.

One of the simplest possible types of two-parameter distributions, aside from the trivial case $f(s, \varphi) = 1$ when $F(r, \theta) = 2\pi$, is that in which $f(s) = 0$ for $s < 1$ and $f(s) = k$ for $s > 1$ (fig. 5).

Here it is readily found from (1) that the corresponding density function is

$$F(r, \theta) = \begin{cases} 0 \text{ for } r < 1 \\ 4\pi k\left(\frac{1}{2} - \frac{1}{\pi} \sin^{-1}\frac{1}{r}\right) \text{ for } r > 1. \end{cases}$$

When rectilinear erasures are not allowed (First Problem) f must be positive or zero everywhere; if we allow rectilinear erasures after the drawing has been made (Second Problem), the restriction that f is of one sign is abandoned; and if we allow only a single uniform erasure (Third Problem), we are interested in the positive solutions $f(s, \varphi)$ obtained when $F(r, \theta)$ is increased by a suitable positive constant.

4. THE SYMMETRIC CASE

A very interesting special case is that in which the density function $F(r, \theta)$ depends merely on the distance from a fixed point, say the origin O, so that $F(r, \theta) = F(r)$ with $F(-r) = F(r)$. This will be called the symmetric case for obvious reasons; the case just specified is of this type. It is natural to conjecture that in the symmetric case we may restrict attention to two-parameter distribution functions $f(s, \varphi)$ of like symmetric type: $f(s, \varphi) = f(s)$ with $f(-s) = f(s)$, i.e., corresponding to lines equally distributed in all directions about the origin.

The fact that we need only consider distribution functions of this symmetric type may be proved as follows.

Suppose first that a non-symmetrical continuous distribution function $f(s, \varphi)$, actually involving φ, does yield a symmetrical density function $F(r)$. It is then clear that $f(s, \varphi+c)$ for any c would yield the same density function, and thus that the average

$$[f(s, \varphi)+f(s, \varphi+\Delta\varphi)+\cdots+f(s, \varphi+(n-1)\Delta\varphi)]/n$$

with $n\Delta\varphi=2\pi$, would also. Proceeding to the limit, we see that the symmetrical distribution function

$$\bar{f}(s)=\int_0^{2\pi} f(s, \varphi)d\varphi/2\pi$$

would also yield the assigned density $F(r)$.

Hence if a non-symmetrical $f(s, \varphi)$ did exist, so would a symmetrical $\bar{f}(s)$.

As follows from a later result, there cannot exist such a non-symmetrical $f(s, \varphi)$. In fact if there did exist such an $f(s, \varphi)$, the difference $g(s, \varphi)=f(s, \varphi)-\bar{f}(s)$ would satisfy the homogeneous linear integral equation,

$$0=\int_0^{2\pi} g(r \sin(\varphi-\theta), \varphi)d\varphi.$$

If we write here $\varphi=\theta+\psi$, this equation takes the equivalent form

$$0=\int_0^{2\pi} g(r \sin\psi, \theta+\psi)d\psi.$$

But it will be proved that the basic equation (1) admits of a unique solution f at most; hence in the case $F=0$, the only possible one is trivial one $f=0$, and this is the equation under consideration with $f=g$. We observe also that if other solutions existed it would be possible to make a rectilinear drawing (non-uniform) and erase it all by rectilinear erasure along other lines. At present, however, we shall only prove that at most one symmetrical solution can exist. Otherwise we obtain the following still simpler equation:

$$\int_0^{2\pi} g(r \sin \psi)d\psi = 0$$

Thus we infer that the integral last written vanishes identically in r. Now introduce the variable $s = r \sin \psi$ in this integral. In this way we conclude that we must have

$$\int_0^r \frac{g(s)ds}{\sqrt{r^2 - s^2}} = 0.$$

This is a special case of a type of integral equation treated by Abel to which we shall refer later. Let us solve it explicitly by his simple, direct method, of which the generalization is immediate. Multiply this integral through by $r/\sqrt{\rho^2 - r^2}$ where $s < r < \rho$, and integrate as to r from 0 to ρ. We obtain thus

$$\int_0^\rho \left[\int_0^r \frac{rg(s)ds}{\sqrt{(\rho^2 - r^2)(r^2 - s^2)}} \right] dr = 0.$$

Making a valid interchange of the order of integration (see fig. 6) this becomes

$$\int_0^\rho g(s) \left(\int_s^\rho \frac{rdr}{\sqrt{(\rho^2 - r^2)(r^2 - s^2)}} \right) ds = 0.$$

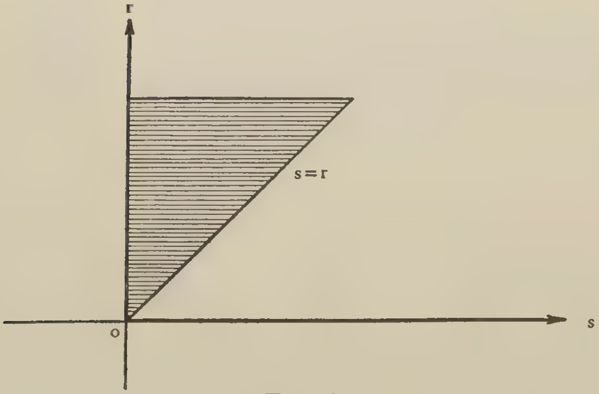


FIG. 6

But the inner definite integral has the value $\pi/2$. Hence we deduce

$$\int_0^\rho g(s)ds=0;$$

and by differentiating as to ρ we conclude for all ρ and χ that $g(\rho)$ is zero. Thus g vanishes identically, contrary to assumption.

We see therefore that in the case of a continuous symmetrical density function $F(r)$ there is at most one corresponding continuous distribution function, f, which, if it exists, must likewise be symmetrical.

In consequence in the symmetric case we may take (1) in the more special form

$$F(r)=\int_0^{2\pi} f\big(r\sin(\varphi-\theta)\big)d\varphi=\int_0^{2\pi} f(r\sin\varphi)d\varphi,$$

where $F(-r)=F(r), f(-s)=f(s)$. Thus we need only to consider the equation

$$F(r)=4\int_0^{\frac{\pi}{2}} f(r\sin\varphi)d\varphi,$$

or, making the change of variables $r\sin\varphi=s$,

$$F(r)=4\int_0^r \frac{f(s)ds}{\sqrt{r^2-s^2}}.$$

But this is precisely the integral equation solved by Abel (1828). In order to solve it we have only to multiply through by $r/\sqrt{\rho^2-r^2}$ and integrate in r from 0 to ρ. This yields, as in the special case treated above,

$$2\pi\int_0^\rho f(s)ds=\int_0^\rho \frac{rF(r)dr}{\sqrt{\rho^2-r^2}}.$$

If there is a continuous solution, the integral on the right must have a continuous derivative in ρ, namely $f(\rho)$.

Thus there exists such a continuous (symmetric) distribution function $f(s)$ if and only if the integral

$$I(s)=\int_0^s \frac{rF(r)dr}{\sqrt{s^2-r^2}}$$

admits a continuous derivative, in which case we have necessarily

$$f(s)=\frac{1}{2\pi}\frac{d}{ds}\int_0^s \frac{rF(r)dr}{\sqrt{s^2-r^2}} \tag{2}$$

as the unique solution.

If we do not wish to restrict attention to continuous functions F and f, various extensions are clearly possible. One of the simplest of these would be that in which F is taken integrable in the sense of Lebesgue over the given region, while f is similarly integrable. Here we should be led to require that the integral $I(s)$ not only exist but be absolutely continuous.

5. A SPECIAL SYMMETRIC CASE

We shall apply the preceding formal work to the discussion of a particular symmetric case which is especially interesting, namely,

$$F(r)=\begin{cases}0 \text{ for } r<r_0\\ k \text{ for } r>r_0.\end{cases}$$

Here the function $F(r)$ is discontinuous at $r=r_0$ but not in a way such as to cause essential difficulty. In fact if we apply the formula (2) to this case we find at once

$$f(s)=\begin{cases}0 \text{ for } s<r_0\\ \dfrac{k}{2\pi}\quad\dfrac{s}{\sqrt{s^2-r_0^2}} \text{ for } s>r_0.\end{cases}$$

Since the function $f(s)$ so obtained is everywhere positive and leads to no difficulty in (1), we conclude that it is possible to set up a symmetrical distribution of lines outside of the circle $s=r_0$ in such wise that the region outside of the given circle is of a *uniform gray*.

In the accompanying drawing (fig. 7) Mr. Middleton takes only sixteen lines for each radial direction and lets the angle increase successively by 5°. His result is shown herewith and evidently accomplishes the desired result as far as could be expected in view of the relatively few lines used for each direction.

If, on the other hand, we take a good many lines for each direction, these directions being at a considerable angle (say $\pi/4$ radians) apart, another interesting type of approximation is obtained (fig. 8).

Since a circular white spot is the "negative" of a dot, it is clear that we can draw a "faint negative" of any drawing by a process of stippling. More precisely, we can reproduce a doubly exposed negative obtained from a first weak exposure to a stippled reproduction of the given drawing and a subsequent strong exposure to uniform light.

6. THE GENERAL HARMONIC CASE

Imagine now a continuous distribution function $f(s, \varphi)$ which is a simple harmonic function of φ of period $\frac{2\pi}{m}$:

$$f_m(s, \varphi) = f_m(s) \cos m\varphi + g_m(s) \sin m\varphi.$$

Here $f_m(s)$, $g_m(s)$ are to be regarded as even or odd according as m is even or odd, so that the functional identity $f_m(-s, \varphi+\pi) = f_m(s, \varphi)$ holds and $f_m(0) = g_m(0) = 0$.

We shall speak of such an $f_m(s, \varphi)$ as harmonic of the mth order ($m \geqq 0$); in particular, the symmetric case is the harmonic case of zero-th order.

A density function $F(r, \theta)$ will similarly be said to be harmonic of the mth order if

$$F(r, \theta) = F_m(r) \cos m\theta + G_m(r) \sin m\theta.$$

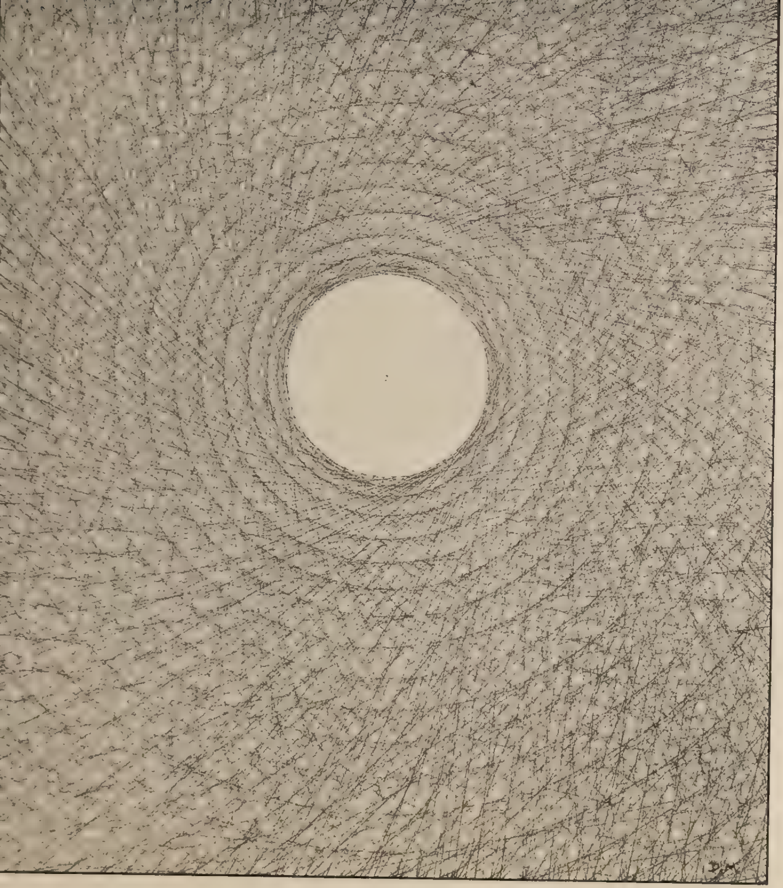
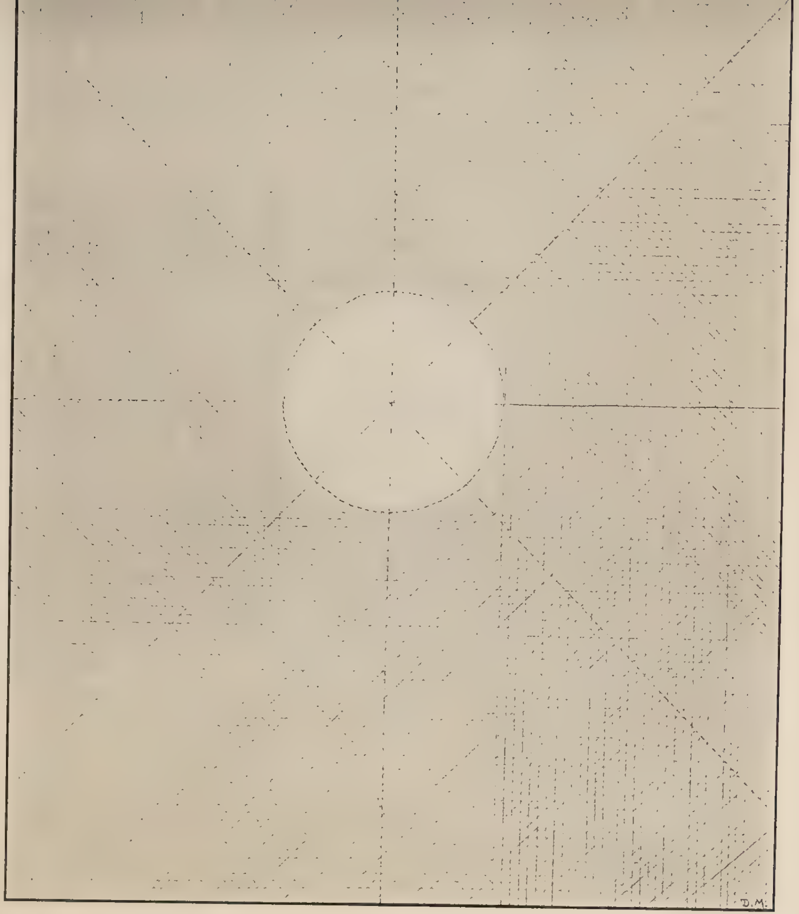


Fig. 7



It is easily established that such a continuous harmonic distribution function of the mth order $f(s, \varphi)$ yields always a corresponding continuous harmonic distribution function of the mth order, $F(r, \theta)$.

In fact, from (1) we obtain directly

$$\begin{aligned} F(r, \theta) = \int_0^{2\pi} & (f_m(r \sin (\varphi - \theta)) \cos m\varphi \\ & + g_m(r \sin (\varphi - \theta)) \sin m\varphi) d\varphi \\ = \int_0^{2\pi} & (f_m(r \sin \chi) \cos m(\chi + \theta) \\ & + g_m(r \sin \chi) \sin m(\chi + \theta)) d\chi \\ = \Big[\int_0^{2\pi} & (f_m(r \sin \chi) \cos m\chi \\ & + g_m(r \sin \chi) \sin m\chi) \, d\chi \Big] \cos m\theta \\ + \Big[\int_0^{2\pi} & (-f_m(r \sin \chi) \sin m\chi \\ & + g_m(r \sin \chi) \cos m\chi) \, d\chi \Big] \sin m\theta. \end{aligned}$$

Conversely we can at least conclude that if for a given harmonic density function $F_m(r, \theta)$ of order m there is a corresponding continuous distribution function $f(s, \varphi)$, this will necessarily be unique and also harmonic of order m.

To establish this uniqueness, suppose that $f(s, \varphi)$ is a distribution function yielding the harmonic $F_m(r, \theta)$ as corresponding density function. The expansion of $f(s, \varphi)$ in a Fourier series breaks $f(s, \varphi)$ up (formally) into an infinite number of harmonic components of orders $0, 1, 2, \cdots$, which, by what has just been proved, are carried over into the Fourier components of $F(s, \varphi)$ of the same orders. Consequently if we can prove that a harmonic distribution function can only be carried into the density function 0 if the distribution function itself vanishes identically, we will have proved that $f(s, \varphi)$ is harmonic of the order m in question. Furthermore it will follow that the function $f(s, \varphi)$, if it exists, is unique.

Thus we need only prove that if a harmonic distribution function of order m, $f_m(s, \varphi)$, is not identically zero it cannot yield a harmonic density function $F_m(r, \theta)$ which is identically zero. This has already been proved for $m=0$. But for $m>0$ we see that $F_m(r, \theta)=0$ would imply

$$\int_0^{2\pi} f_m(r \sin \chi) \cos m\chi d\chi = \int_0^{2\pi} g_m(r \sin \chi) \sin m\chi d\chi = 0$$

for $f_m(s)$ and $g_m(s)$ not both identically zero.

We will only prove this to be impossible for the cases $m=1,2$ since the extension can then be made at once to the cases $m=3,4 \cdots$. Suppose that we have for $m=1$

$$\int_0^{2\pi} f_1(r \sin \chi) \cos \chi d\chi = 4\int_0^{\frac{\pi}{2}} f_1(r \sin \chi) \cos \chi d\chi = 0.$$

Multiply through by r and integrate; we obtain at once $f_1^{(-1)}(r)=0$, where we write

$$f_1^{(-1)}(u) = \int_0^u f_1(u) du.$$

Hence we infer $f_1(r) \equiv 0$. Likewise if we have

$$\int_0^{\frac{\pi}{2}} g_1(r \sin \chi) \sin \chi d\chi = 0,$$

then by multiplying through by dr and integrating from 0 to r,

$$\int_0^{\frac{\pi}{2}} g_1^{(-1)}(r \sin \chi) d\chi = 0.$$

This is a homogeneous equation of the Abel type in $g_{1^{(-1)}}(s)$, and we find similarly

$$g_1^{(-1)}(s) \equiv 0, \text{ and so } g_1(s) \equiv 0.$$

We pass now to the case $m=2$. Suppose that we have

$$\int_0^{2\pi} f_2(r \sin \chi) \cos 2\chi d\chi = 0,$$

which we write in the form

$$\int_0^{2\pi} f_2(r \sin \chi)(\cos \chi \cos \chi - \sin \chi \sin \chi)d\chi = 0.$$

On integrating by parts the first term evidently takes the form

$$\frac{1}{r}\int_0^{2\pi} f_2^{(-1)}(r \sin \chi) \sin \chi d\chi,$$

while the second is seen to be equal to

$$-\frac{d}{dr}\int_0^{2\pi} f_2^{(-1)}(r \sin \chi) \sin \chi d\chi.$$

Hence if we write

$$W = \int_0^{2\pi} f_2^{(-1)}(r \sin \chi) \sin \chi d\chi,$$

we conclude that

$$\frac{1}{r}W - W' \equiv 0,$$

i.e., $W = cr$. But since $W'(0) = 0$ we must have $c = 0$, and so $W \equiv 0$.

Employing now the same argument as in the case $m = 1$, for the equation $W = 0$ we find that $f_2^{(-1)}(s) = 0$ and so $f_2(s) \equiv 0$.

Likewise we readily infer that $g_2(s) \equiv 0$.

More generally, we may obtain the stated result for $m = 1, 2 \cdots$ in succession, by writing at the mth stage

$$\cos m\chi = \cos \chi \cos (m-1)\chi - \sin \chi \sin (m-1)\chi$$
$$\sin m\chi = \sin \chi \cos (m-1)\chi + \cos \chi \sin (m-1)\chi$$

in the equations involving $f_m(s)$ and $g_m(s)$ respectively, and thus reducing the question to one of the type considered at the $(m-1)$st stage.

In this manner the stated result is readily established.

We are now in a position to formulate preliminary necessary and sufficient conditions for a continuous solution of the general harmonic case of the mth order.

A necessary and sufficient condition that, to a general harmonic continuous density function of the mth order,

$$F(r, \theta) = F_m(r) \cos m\theta + G_m(r) \sin m\theta,$$

there corresponds a continuous distribution function [unique and harmonic of the same order m by what precedes],

$$f(s, \phi) = f_m(s) \cos m\phi + g_m(s) \sin m\phi$$

is that the integral equation for $h_m(s)$,

(3) $$H_m(r) = \int_0^{2\pi} h_m(r \sin u) e^{-imu} du$$

where

$$H_m(r) = F_m(r) + iG_m(r),$$

admits of a continuous solution, whose real and imaginary coefficients will then yield $f_m(s)$ *and* $g_m(s)$ *respectively.*

The proof of this italicized statement follows immediately from (1) and what has been proved in the preceding section.

7. EXPLICIT SOLUTION IN THE HARMONIC CASE $m=1$

In the harmonic case $m = 1$ we have to consider the integral equation

(3_1) $$H_1(r) = \int_0^{2\pi} h_1(r \sin u) e^{-iu} du,$$

which expresses the relation between the known coefficients $F_1(r)$, $G_1(r)$ of the density function F and the like coefficients $f_1(s)$, $g_1(s)$ of the distribution function $f(s, \phi)$ which it is desired to find. This equation may be written

$$H_1(r) = \int_0^{2\pi} h(r \sin u)(\cos u - i \sin u) du.$$

But the first component of the integral on the right is clearly

$$\frac{1}{r} h_1^{(-1)} (r \sin u) \Big|_0^{2\pi} = 0,$$

while the second term evidently has the value

$$-i\frac{d}{dr}\int_0^{2\pi} h_1^{(12)}(r\sin u)du = -4i\frac{d}{dr}\int_0^{\frac{\pi}{2}} h^{(-1)}(r\sin u)du.$$

Thus the equation under consideration is equivalent to

$$\frac{i}{4}\int_0^r H_1(r)dr = \int_0^{\frac{\pi}{2}} h^{(-1)}(r\sin u)du,$$

at least if the function $H_1(r)$ is suitably restricted near $r=0$. For example, it would suffice if we assumed that $H_1(r)$ remains finite near $r=0$. But the right-hand member is evidently the same as

$$\int^r h_1^{(-1)}(s)\frac{ds}{\sqrt{r^2-s^2}},$$

so that the equation is essentially a linear integral equation of Abel type for $h_1^{(-1)}(s)$. Hence we obtain

$$h_1^{(-1)}(s) = \frac{i}{4\pi}\frac{d}{ds}\left[\int_0^s \frac{r\left(\int_0^r H_1(r_1)dr_1\right)dr}{\sqrt{s^2-r^2}}\right],$$

where the outer integral in the right-hand member clearly vanishes for $s=0$ and has a continuous first derivative. Hence this equation is equivalent to

$$h_1(s) = \frac{i}{2\pi}\frac{d^2}{ds^2}\left[\int_0^s \frac{r\left(\int_0^r H_1(r_1)dr_1\right)dr}{\sqrt{s^2-r^2}}\right].$$

Thus we infer that a necessary and sufficient condition for a solution to exist is that the integral on the right side of the equation just written admits a continuous second derivative, in which case the unique solution is provided by the same equation.

But the double integral on the right may clearly be written in inverse order of integration as

$$\int_0^s H_m(t)\left(\int_t^s \frac{rdr}{\sqrt{s^2-r^2}}\right)dt = \int_0^s H_m(t)(s^2-t^2)^{\frac{1}{2}}dt,$$

so that

$$h_1(s) = \frac{i}{2\pi} \frac{d^2}{ds^2}\left[\int_0^s H_1(t)(s^2 - t^2)^{\frac{1}{2}} dt\right].$$

Further, we may differentiate once *under* the integral sign so that $(s^2 - t^2)^{\frac{1}{2}}$ is replaced by the derivative of this expression. If we write $s = tu$ we see, however, that if $u = s/t > 1$,

$$\frac{d}{ds}(s^2 - t^2)^{\frac{1}{2}} = \frac{d}{du}(u^2 - 1)^{\frac{1}{2}}.$$

Hence we obtain the final solution for $m = 1$:

$$h_1(s) = \frac{i}{4\pi} \frac{d}{ds}\left[\int_0^s H_1(t) \frac{2u}{\sqrt{u^2 - 1}} dt\right], \quad \left(u = \frac{s}{t} > 1\right). \tag{4}$$

Thus our general conclusion in the case of a harmonic distribution of the first order is as follows: *In the case of a given harmonic density function of the first order* $(m = 1)$

$$F_1(r, \theta) = F_1(r) \cos \theta + G_1(r) \sin \theta,$$

where $H_1(r) = F_1(r) + i G_1(r)$ *remains finite near* $r = 0$,[3] *there exists a corresponding continuous distribution function,*

$$f_1(s, \varphi) = f_1(s) \cos \varphi + i\, g_1(s) \sin \varphi,$$

likewise bounded near $s = 0$ *and harmonic of the first order, if and only if the integral in* (4) *represents a function with a continuous derivative. In this event the unique solution* $h_1(s)$ *is exhibited in* (4), *where of course,*

$$h_1(s) = f_1(s) + i\, g_1(s).$$

9. THE SOLUTION IN HIGHER HARMONIC CASES $m > 1$

The method of solution used above applies essentially to any order $m > 1$. Here we start by writing the identity

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[3] We are assuming that $F(r, \theta)$ is continuous for $r > 0$ only.

$$e^{-im\varphi} = (\cos\varphi - i\sin\varphi)e^{-i(m-1)\varphi},$$

and then proceeding essentially as in the case $m=1$ treated above.

We therefore content ourselves with stating the final result and giving in an appended Note the series of reductions involved. The reader who wishes a more detailed deduction will find it in my article referred to at the outset.

In the case of a given harmonic distribution function of the mth order,

$$F_m(r, \theta) = F_m(r)\cos m\theta + G_m(r)\sin m\theta,$$

where $H_m(r) = F_m(r) + iG_m(r)$ *is supposed to vanish to the* $(m-1)$*st order at* $r=0$ *so that* $H_m(r)/r^{m-1}$ *remains finite, there exists a corresponding continuous distribution function.*

$$f_m(s, \varphi) = f_m(s)\cos m\varphi + g_m(s)\sin m\varphi,$$

likewise harmonic of the mth order, if and only if the integral in the equation

$$(5)\quad h_m(s) = \frac{i^m}{4\pi}\frac{d}{ds}\int_0^s H_m(t)\frac{(u+\sqrt{u^2-1})^m + (u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}\,dt,$$

$$\left(u = \frac{s}{t} > 1\right),$$

is continuous and admits a continuous derivative as to s, *in which case the unique solution is derived by setting* $h_m(s)$ *in* (5) *equal to* $f_m(s) + ig_m(s)$.

10. EXPLICIT SOLUTION IN THE NON-HARMONIC CASE

In virtue of the known relationship between continuous functions and their Fourier series we are now in a position to formulate the general solution of our problem:

If the given continuous density function $F(r, \theta)$ *be expanded in a Fourier series*

$$\tfrac{1}{2}F_o(r)+\sum_{m=1}^{\infty}(F_m(r)\cos m\theta+G_m(r)\sin m\theta),$$

and if $F_m(r)$ *and* $G_m(r)$ *vanish to the*$(m\text{-}1)$*th order in* r *for all* m,[4] *then there will exist a corresponding continuous distribution function if and only if the integrals*

$$I_m(s)=\int_0^s(F_m(t)+iG_m(t))\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}dt \qquad \left(u=\frac{s}{t}>1\right), \tag{6}$$

are continuous for $s\geqq 0$ *together with their first derivatives in such wise that the continuous functions* $f_m(s)$, $g_m(s)$ *defined by*

$$f_m(s)+ig_m(s)=\frac{i^m}{4\pi}I'_m(s) \tag{7}$$

form the Fourier coefficients of a continuous function $f(s,\varphi)$ *corresponding to*

$$\tfrac{1}{2}f_o(s)+\sum_{m=1}^{\infty}(f_m(s)\cos m\varphi+g_m(s)\sin m\varphi).$$

In this case $f(s,\varphi)$ *forms the unique continuous solution of the distribution problem.*

The proof is readily made by means of an induction based on the equations of the Note appended to the present paper.[5]

II. ON THE THREE DRAWING PROBLEMS

On the basis of what precedes we obtain immediately the following general conclusions for the three drawing problems specified earlier:

(1) A drawing with given continuous[6] density function $F(r,\theta)\geqq 0$ can be made without rectilinear erasures by means

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[4] This condition will be satisfied, for instance, if $F(r,\theta)$ has continuous partial derivatives in x and y of all orders k in open continua S_k containing the origin.

[5] See also my paper already referred to.

[6] That is, continuous in x and y.

of a suitable continuous[7] two-parameter distribution function $f(s, \phi)$ if and only if $f(s, \varphi)$ exists as specified above and is positive or zero. If such a solution exists it will be unique.

(2) A drawing with given density function $F(r, \theta) \geqq 0$ can be made by means of a continuous two-parameter continuous distribution function $f_1(s, \varphi) \geqq 0$, followed by a similar distribution function $f_2(s, \varphi)$ of non-overlapping rectilinear erasures, if and only if $f(s, \varphi)$ exists as specified above, when we may take

$$f_1(s, \varphi) = \big(|f(s, \varphi)| + f(s, \varphi)\big)/2, \; f_2(s, \varphi) = \big(|f(s, \varphi)| - f(s, \varphi)\big)/2$$

so that

$$f(s, \varphi) = f_1(s, \varphi) - f_2(s, \varphi).$$

If such a solution exists it is unique, and evidently requires the least possible drawing and subsequent erasure.[8]

(3) A drawing with given continuous density function $F(r, \theta) \geqq 0$ can be made by means of a continuous two-parameter distribution function $f(s, \varphi)$ followed by a single *uniform* erasure if and only if $f(s, \varphi)$ exists as specified above. If $f(s, \varphi)$ is positive or zero everywhere no subsequent erasure is of course necessary. If, however, $f(s, \varphi)$ has a negative minimum $-m$, we first make the drawing with positive density function $F(r, \theta) + 2\pi m$ and corresponding distribution function $f(s, \varphi) + m \geqq 0$, and then make a uniform erasure with density $2\pi m$. Evidently this solution is essentially unique, and requires the least possible uniform erasure.

It is interesting to note that in the first problem when no erasures are allowed, certain obvious geometrical conditions must be satisfied if the drawing is to be possible. In order to illustrate this fact as simply as possible let us restrict attention to the symmetric case, $F = F(r)$, when, as we have seen, the distribution function must also be symmetric, $f = f(s)$.

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[7] That is, continuous in s and φ.

[8] I.e., as measured by lead put down and lead erased.

If we consider two circles C_a and C_b of radius a, b with $a > b > 0$, concentric with the origin it is clear that the ratio of the length of any chord of C_a to the length of the part of the chord within C_b is at least a to b. It follows that the amount of lead $2\pi\iint F(r, \theta) r dr d\theta$ within C_a is at least a/b times that within C_b, so that we must have

$$\frac{1}{a}\int_0^a F(r) r dr \geqq \frac{1}{b}\int_0^b F(r) r dr \qquad (a > b),$$

if the drawing is to be possible without rectilinear erasures. This conclusion may of course also be derived directly from the fundamental equation (1) above, under the assumption $f(s, \varphi) \geqq 0$, the proof involving nothing more than an application of the geometric fact about chords stated above.

APPENDED NOTE

To treat the general case we start from the equations

$$(1) \qquad H_m(r) = \int_0^{2\pi} h_m(r \sin u) e^{-imu} du \qquad (m = 0, 1, 2, \cdots)$$

where the $H_m(r)$ are assumed to be continuous for $r > 0$ and vanish with r to the $(m-1)$st order. These equations may be written

$$(2) \qquad \begin{aligned} H_m(r) &= \int_0^{2\pi} h_m(r \sin u)\,(\cos u - i \sin u) e^{-i(m-1)u} du \\ &= \frac{i(m-1)}{r}\int_0^{2\pi} h_m^{(-1)}(r \sin u) e^{-i(m-1)u} du \\ &\quad - i\frac{d}{dr}\left[\int_0^{2\pi} h_m^{(-1)}(r \sin u) e^{-i(m-1)u} du\right], \end{aligned}$$

where we define

$$(3) \qquad h_m^{(-1)}(s) = \int_0^s h_m(s) ds, \quad h_m^{(-2)}(s) = \int_0^s h_m^{(-1)}(s) ds, \cdots$$

If we multiply through by ir^{-m+1} and integrate from 0 to r, there results

$$(1') \qquad ir^{m-1}\int_0^r \frac{H_m(t)}{t^{m-1}}dt = \int_0^{2\pi} h_m^{(-1)}(r\sin u)e^{-i(m-1)u}du,$$

since the right-hand member vanishes to order m at least in r. But (1′) is of a form like (1) on the right except that m is replaced by $m-1$. Repeating the same type of procedure $m-1$ times, we obtain finally

$$(4) \qquad i^m\int_0^r\left[\int_0^{t_m} t_{m-1}\cdots\left(\int_0^{t_1}\frac{H_m(t)}{t^{m-1}}dt\right)\cdots\right]dt_m = \int_0^{2\pi} h_m^{(-m)}(r\sin u)du.$$

If now we replace the m-fold integral on the left by a simple integral, (4) becomes

$$(5) \qquad \frac{2\left(\frac{i}{2}\right)^m}{(m-1)!}\int_0^r H_m(t)(r^2-t^2)^{m-1}dt = \int_0^{2\pi} h_m^{(-m)}(r\sin u)du.$$

But $h_m^{(-m)}(r)$ is even in r so that the integral on the right in (4) is four times the same integral taken between 0 and $\pi/2$. Writing $s=r\sin u$ we thus obtain, as the equivalent of (4), a linear integral equation of Abel type in $h_m^{(-m)}(s)$ with explicit solution

$$h_m(s) = \frac{1}{\pi}\,\frac{\left(\frac{i}{2}\right)^m}{(m-1)!}\,\frac{d^{m+1}}{ds^{m+1}}\left[\int_0^r r\left(\int_0^r\frac{H_m(t)}{t^{m-1}}(r^2-t^2)^{m-1}dt\right)\frac{dr}{\sqrt{s^2-r^2}}\right].$$

Conversely, if $h_m(s)$ as thus defined exists and is continuous, the equation (1) will be satisfied by $h_m(s)$.

But the right-hand integral, after inverting the order of integration, becomes

$$\int_{\circ}^{s}\frac{H_m(t)}{t^{m-1}}\left[\int_{t}^{s}\frac{(r^2-t^2)^{m-1}}{\sqrt{s^2-r^2}}rdr\right]dt,$$

so that we have

$$(6)\quad h_m(s)=\frac{1}{2\pi}\,\frac{i^m}{1\cdot 3\cdot\cdots\cdot 2m-1}\,\frac{d^{m+1}}{dr^{m+1}}\int_{\circ}^{s}\frac{H_m(t)}{t^{m+1}}(s^2-t^2)^{m-\frac{1}{2}}\,dt.\Big]$$

Now we may differentiate m times as to s *under* the integral sign because the integral and its first $m-1$ derivatives vanish for $t=s$. Writing then $s=tu$ in that $u\geqq 1$ we obtain the equivalent form

$$(6')\quad h_m(s)=\frac{1}{2\pi}\,\frac{i^m}{1\cdot 3\cdots 2m-1}\,\frac{d}{ds}\int_{\circ}^{s}H_m(t)\frac{d^m}{du^m}(u^2-1)^{m-\frac{1}{2}}dt.$$

But it is readily proved by induction that

$$(7)\quad \frac{d^{m-1}}{du^{m-1}}(u^2-1)^{m-\frac{1}{2}}=$$
$$\frac{1\cdot 3\cdot\cdots\cdot 2m-1}{2m}\left[(u+\sqrt{u^2-1})^m-(u-\sqrt{u^2-1})^m\right].$$

Substituting, we obtain the stated final explicit formula

$$(8)\quad h_m(s)=\frac{i^m}{4\pi}\frac{d}{ds}\left[\int_{\circ}^{s}H_m(t)\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}dt\right]$$
$$\left(u=\frac{s}{t}>1\right)$$

yielding the explicit Fourier coefficient $f_m(s)$, $g_m(s)$ for $f(s,\varphi)$ in virtue of the formula $h_m(s)=f_m(s)+ig_m(s)$.

GEORGE D. BIRKHOFF.

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